Atlantic Canada Mathematics Curriculum

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CURRICULUM

Geometry and Applications in Mathematics 111/112

(Implementation Edition)

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- The teachers and other educators and stakeholders across Atlantic Canada who contributed to the development of the *Geometry and Applications in Mathematics 111/112* curriculum guide.

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I. Background and Rationale

A. Background

Mathematics curriculum reform in Atlantic Canada is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society. Curriculum reform has been motivated by a desire to ensure that students in Atlantic Canada benefit from world-class curriculum and instruction in mathematics as a significant part of their learning experiences.

The *Foundation for the Atlantic Canada Mathematics Curriculum* (1996) firmly establishes the *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) as a guiding beacon for pursuing this vision, which embraces the principles of students learning to value and become active "doers" of mathematics and advocates a curriculum which focusses on the unifying ideas of mathematical problem solving, communication, reasoning, and connections. These principles and unifying ideas are reaffirmed with the publication of NCTM's *Principles and Standards for School Mathematics* (2000). The *Foundation for the Atlantic Canada Mathematics Curriculum* establishes a framework for the development of detailed grade-level documents describing mathematics curriculum and guiding instruction.

Mathematics curriculum development has taken place under the auspices of the Atlantic Provinces Education Foundation (APEF), an organization sponsored and managed by the governments of the four Atlantic Provinces. APEF has brought together teachers with department of education officials to co-operatively plan and execute the development of curricula in mathematics, science, language arts, and other curricular areas. Each of these curriculum efforts has been aimed at producing a program that would ultimately support the Essential Graduation Learnings (EGLs), also developed regionally. These EGLs and the contribution of the mathematics curriculum to their achievement are presented in the "Outcomes" section of the mathematics foundation document.

B. Rationale

The *Foundation for the Atlantic Canada Mathematics Curriculum* provides an overview of the philosophy and goals of the mathematics curriculum, presenting broad curriculum outcomes and addressing a variety of issues with respect to the learning and teaching of mathematics. This curriculum guide is one of several which provide greater specificity and clarity for the classroom teacher. The *Foundation for the Atlantic Canada Mathematics Curriculum* describes the mathematics curriculum in terms of a series of outcomes—General Curriculum Outcomes (GCOs), which relate to subject strands, and Key-Stage Curriculum Outcomes (KSCOs), which articulate the GCOs further for the end of grades 3, 6, 9, and 12. This guide builds on the structure introduced in the foundation document, by relating Specific Curriculum Outcomes (SCOs) to KSCOs for *Geometry and Applications in Mathematics 111/112*. Figure 1 further clarifies the outcome structure.

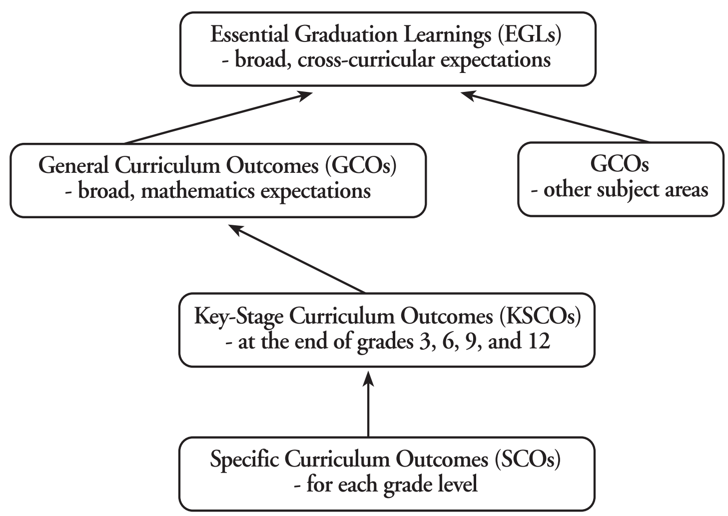


Figure 1: Outcome Framework

This mathematics guide is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice, including the following: (i) mathematics learning is an active and constructive process; (ii) learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates; (iii) learning is most likely when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and nurtures positive attitudes and sustained effort; (iv) learning is most effective when standards of expectation are made clear and assessment and feedback are ongoing; and (v) learners benefit, both socially and intellectually, from a variety of learning experiences, both independent and in collaboration with others.

II. Program Design and Components

A. Program Organization

As indicated previously, the mathematics curriculum is designed to support the Atlantic Canada Essential Graduation Learnings (EGLs). The curriculum is designed to significantly contribute to students meeting each of the six EGLs, with the communication and problem-solving EGLs relating particularly well with the curriculum's unifying ideas. (See the "Outcomes" section of the *Foundation for the Atlantic Canada Mathematics Curriculum*.) The foundation document then goes on to present student outcomes at key stages of the student's school experience.

This curriculum guide presents specific curriculum outcomes for *Geometry and Applications in Mathematics 111/112*. As illustrated in Figure 2, these outcomes represent the step-by-step means by which students work toward accomplishing the key-stage curriculum outcomes, the general curriculum outcomes, and, ultimately, the essential graduation learnings.

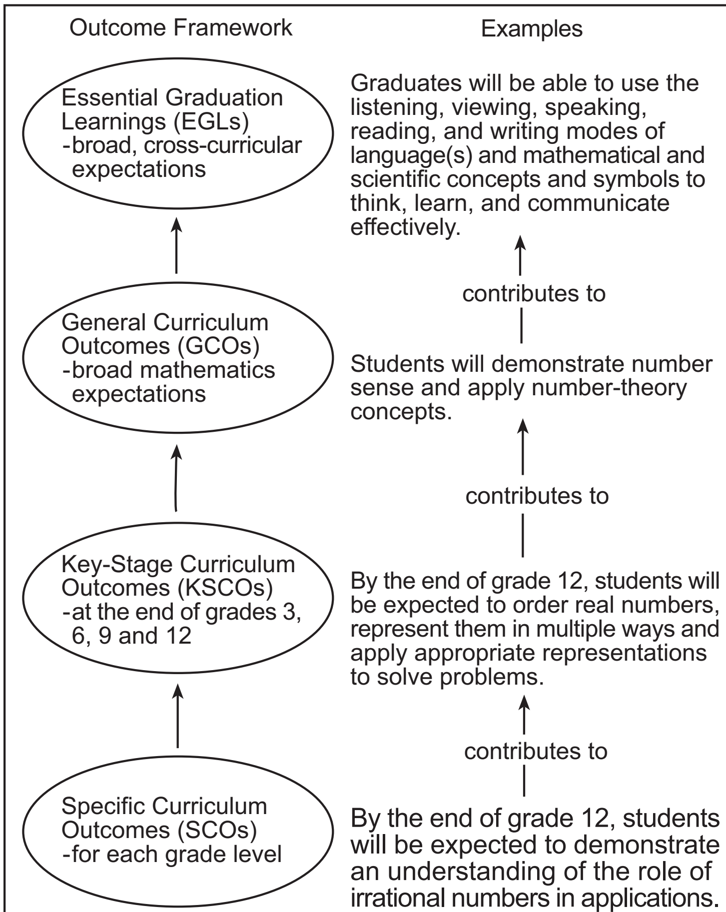


Figure 2: Examples of Outcomes

It is important to emphasize that the initial presentation of the specific curriculum outcomes for this course (pp. 17-32) follows the outcome structure established in the *Foundation for the Atlantic Canada Mathematics Curriculum* and does not represent a natural teaching sequence. In *Geometry and Applications in Mathematics 111/112*, however, a suggested teaching order for specific curriculum outcomes has been given within a sequence of four topics or units (i.e., Statistics; Independent Study; Probability; and Circle Geometry). While the units are presented with a specific teaching sequence in mind, some flexibility exists as to the ordering of units within the course. It is expected that teachers will make individual decisions as to what sequence of topics will best suit their classes. In most instances, this will occur in consultation with fellow staff members, department heads, and/or district level personnel.

Decisions on sequencing will depend on a number of factors, including the nature and interests of the students themselves. For instance, what might serve well as a "kickoff" topic for one group of students might be less effective in that role with a second group. Another consideration with respect to sequencing will be co-ordinating the mathematics program with other aspects of the students' school experience. An example of such co-ordination would be studying aspects of measurement in connection with appropriate topics in science. As well, sequencing could be influenced by events outside of the school, such as elections, special community celebrations, or natural occurrences.

B. Unifying Ideas

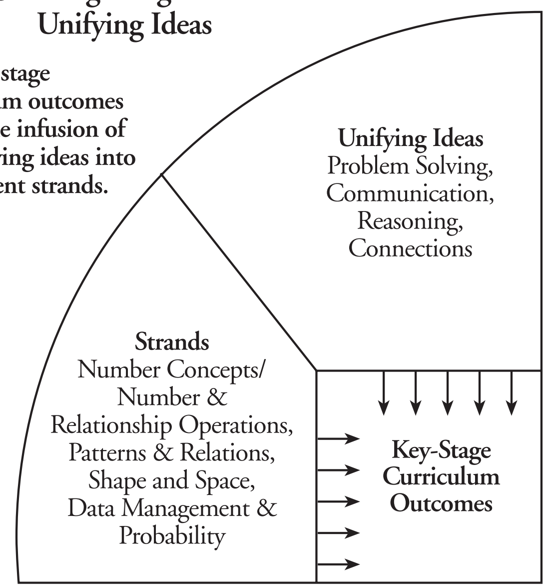
The NCTM *Curriculum and Evaluation Standards* (1989) and *Principles and Standards* (2000) establishes mathematical problem solving, communication, reasoning, and connections as central elements of the mathematics curriculum. The *Foundation for the Atlantic Canada Mathematics Curriculum* (pp. 7-11) further emphasizes these unifying ideas and presents them as being integral to all aspects of the curriculum. Indeed, while the general curriculum outcomes are organized around content strands, every opportunity has been taken to infuse the key-stage curriculum outcomes with one or more of the unifying ideas. This is illustrated in Figure 3.

These unifying concepts serve to link the content to methodology. They make it clear that mathematics is to be taught in a problem-solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them.

Students will be expected to address routine and/or non-routine

Figure 3 - Integrating Strands and Unifying Ideas

The key-stage curriculum outcomes reflect the infusion of the unifying ideas into the content strands.



mathematical problems on a daily basis. Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, strategies such as try-and-adjust, look for a pattern, draw a picture, act it out, use models, make a table or chart, and make an organized list should all become familiar to students during their early years of schooling, whereas working backward, logical reasoning, trying a simpler problem, changing point of view, and writing an open sentence or equation would be part of a student's repertoire in the later elementary years. During middle school and the 9/10 years, this repertoire will be extended to include such strategies as interpreting formulas, checking for hidden assumptions, examining systematic or critical cases, and solving algebraically. *Geometry and Applications in Mathematics 111/112* will continue to develop students' problem-solving reportoires.

Opportunities should be created frequently to link mathematics and career opportunities. Students need to be aware of the importance of mathematics and the need for mathematics in so many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in higher-level mathematics programming in senior high mathematics and beyond.

C. Learning and Teaching Mathematics

The unifying ideas of the mathematics curriculum suggest quite clearly that the mathematics classroom needs to be one in which students are actively engaged each day in the doing of mathematics. No longer is it sufficient or proper to view mathematics as a set of concepts and algorithms for the teacher to transmit to students. Instead, students must come to see mathematics as a vibrant and useful tool for helping them understand their world, and as a discipline which lends itself to multiple strategies, student innovation, and, quite often, multiple solutions. (See the "Contexts for Learning and Teaching" section of the foundation document.)

The learning environment will be one in which students and teachers make regular use of manipulative materials and technology, and actively participate in discourse and conjecture, verify reasoning, and share solutions. This environment will be one in which respect is given to all ideas in which reasoning and sense making are valued above "getting the right answer." Students will have access to a variety of learning resources, will balance the acquisition of procedural skills with attaining conceptual understanding, will estimate routinely to verify the reasonableness of their work, will compute in a variety of ways while continuing to place emphasis on mental computation skills, and will engage in homework as a useful extension of their classroom experiences.

D. Meeting the Needs of All Learners

The *Foundation for the Atlantic Canada Mathematics Curriculum* stresses the need to deal successfully with a wide variety of equity and diversity issues. Not only must teachers be aware of, and adapt instruction to account for, differences in student readiness, but they must also remain aware of avoiding gender and cultural biases in their teaching. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom.

NCTM's *Principles and Standards* (2000) cites equity as a core element of its vision for mathematics education. "All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students"(p. 12).

At grade 11 in New Brunswick, variations in student readiness, aptitude, and post-secondary intentions are addressed in significant part by the provision of courses at levels 1, 2 and 3. Students at all levels will work toward achievement of the same key-stage and general curriculum outcomes, and many of the course-specific curriculum outcomes will also be the same or similar. As well, the instructional environment and philosophy should be the same at all levels, with high expectations maintained for all students. The significant difference between levels will be the depth, breadth, and degree of sophistication and formalism expected with respect to each general outcome. Similarities between courses should allow some students to move from one course level to another.

By and large, Level 3 courses will be characterized by a greater focus on concrete activities, models, and applications, with less emphasis given to formalism, symbolism, computational or symbol-manipulating facility, and mathematical structure. Level 1 and 2 courses will involve greater attention to abstraction and more sophisticated generalizations, while Level 3 courses would see less time spent on complex exercises and connections with advanced mathematical ideas. Level 1 courses, which are designed for particularly talented students of mathematics, will be characterized by both more sophisticated engagement with mathematical concepts and techniques, and the extension of some topics beyond the scope provided at Level 2. These extensions will be included in Level 2 curriculum guides and identified with a *** ** *   * ** *** symbol.

By way of a brief illustration, students at all levels should develop an understanding of exponential relationships. Students taking Level 3 courses have as much need as others to understand the nature of exponential relationships, given the central place of these relationships in universal, everyday issues such as investment, personal and government debt, and world population dynamics. The nature of exponential relationships can be developed through concrete, hands-on experiments and data analysis that does not require a lot of formalism or symbol manipulation. The more formal and symbolic operations on exponential relationships will be much more prevalent in Level 1 and 2 courses.

Finally, within any given course at any level, teachers must understand, and design instruction to accommodate, differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Further, the practice of designing classroom activities to support a variety of learning styles must be extended to the assessment realm; such an extension implies the use of a wide variety of assessment techniques, including journal writing, portfolios, projects, presentations, and structured interviews.

E. Support Resources

This curriculum guide represents the central resource for the teacher of *Geometry and Applications in Mathematics 111/112*. Other resources are ancillary to it. This guide should serve as the focal point for all daily, unit, and course-long planning, as well as a reference point to determine the extent to which the instructional outcomes should be met.

Nevertheless, other resources will be significant in the mathematics classroom. Textual and other print resources will be significant to the extent that they support the curriculum goals. Teachers will need professional resources as they seek to broaden their instructional and mathematical skills. Key among these are the NCTM publications, including the *Principles and Standards for School Mathematics, Assessment Standards for School Mathematics, Curriculum and Evaluation Standards for School Mathematics*, the *Addenda Series*, *Professional Standards for Teaching Mathematics,* and the various NCTM journals and yearbooks. As well, manipulative materials and appropriate access to technological resources (e.g., software, videos) should be available. Calculators will be an integral part of many learning activities.

F. Role of Parents

Societal change dictates that students' mathematical needs today are in many ways different than were those of their parents. These differences are manifested not only with respect to mathematical content, but also with respect to instructional approach. As a consequence, it is important that educators take every opportunity to discuss with parents changes in mathematical pedagogy and why these changes are significant. Parents who understand the reasons for changes in instruction and assessment will be better able to support their children in mathematical endeavours by fostering positive attitudes towards mathematics, stressing the importance of mathematics in their children's lives, assisting children with mathematical activities at home, and, ultimately, helping to ensure that their children become confident, independent learners of mathematics.

G. Connections Across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating learning experiences—through teacher-directed activities, group or independent exploration, and other opportune learning situations. However, it should be remembered that certain aspects of mathematics are sequential, and need to be developed in the context of structured learning experiences.

The concepts and skills developed in mathematics are applied in many other disciplines. These include science, social studies, music, technology education, art, physical education, and home economics. Efforts should be made to make connections and use examples which apply across a variety of discipline areas.

In science, for example, the concepts and skills of measurement are applied in the context of scientific investigations. Statistical concepts and skills are applied as students collect, present, and analyse data. Examples and applications of many mathematical relations and functions abound.

In social studies, knowledge of confidence intervals is valuable in intrepreting polling data, and an understanding of exponential growth is necessary to appreciate the significance of government debt and population growth. As well, students read, interpret, and construct tables, charts, and graphs in a variety of contexts such as demography.

Opportunities for mathematical connections are also plentiful in physical educaiton, many technological courses and the fine arts.

III. Assessment and Evaluation

A. Assessing Student Learning

Assessment and evaluation are integral to the process of teaching and learning. Ongoing assessment and evaluation are critical, not only with respect to clarifying student achievement and thereby motivating student performance, but also for providing a basis upon which teachers make meaningful instructional decisions. (See "Assessing and Evaluating Student Learning" in the *Foundation for the Atlantic Canada Mathematics Curriculum*.)

Characteristics of good student assessment should include the following: i) using a wide variety of assessment strategies and tools; ii) aligning assessment strategies and tools with the curriculum and instructional techniques; and iii) ensuring fairness both in application and scoring. The *Principles for Fair Student Assessment Practices for Education in Canada* elaborate good assessment practice and serve as a guide with respect to student assessment for the mathematics foundation document. (See also, Appendix A, "Assessing and Evaluating Student Learning.")

B. Program Assessment

Program assessment will serve to provide information to educators as to the relative success of the mathematics curriculum and its implementation. It will address such questions as the following: Are students meeting the curriculum outcomes? Is the curriculum being equitably applied across the region? Does the curriculum reflect a proper balance between procedural knowledge and conceptual understanding? Is technology fulfilling a proper role?

IV. Designing an Instructional Plan

It is important to develop an instructional plan for the duration of the course. Without such a plan, it is easy to run out of time before all aspects of the curriculum have been addressed. A plan for instruction that is comprehensive enough to cover all outcomes and topics will help to highlight the need for time management.

It is often advisable to use pre-testing to determine what students have retained from previous grades relative to a given topic or set of outcomes. In some cases, pre-testing may also identify students who have already acquired skills relevant to the current course. Pre-testing is often most useful when it occurs one to two weeks prior to the start of a a topic or set of outcomes. When the pre-test is done early enough and exposes deficiencies in prerequisite knowledge/ skills for individual students, sufficient time is available to address these deficiencies prior to the start of the topic/unit. When the whole group is identified as having prerequisite deficiencies, it may point to a lack of adequate development or coverage in previous grades. This may imply that an adjustment is required to the starting point for instruction, as well as a meeting with other grade level teachers to address these concerns as necessary.

Many topics in mathematics are also addressed in other disciplines, even though the nature and focus of the desired outcome is different. Whenever possible, it is valuable to connect the related outcomes of various disciplines. This can result in an overall savings in time for both disciplines. The most obvious of these connections relate to the use of measurement in science and the use of a variety of data displays in social studies.

V. Curriculum Outcomes

The pages that follow provide details regarding both specific curriculum outcomes and the four topics/units that comprise *Geometry and Applications in Mathematics 111/112*. The specific curriculum outcomes are presented initially, then the details of the units follow in a series of two-page spreads. (See Figure 4 on next page.)

This guide presents the curriculum for *Geometry and Applications in Mathematics 111/112* so that a teacher may readily view the scope of the outcomes which students are expected to meet during the year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings in this course are part of a bigger picture of concept and skill development. (See Appendix B for a complete listing of the SCOs for grades 9 and 10.)

Within each unit, the specific curriculum outcomes are presented on two-page spreads. At the top of each page, the overarching topic is presented, with the appropriate SCO(s) displayed in the left-hand column. The second column of the layout is entitled "Elaboration-Instructional Strategies/Suggestions" and provides a clarification of the specific curriculum outcome(s), as well as some suggestions of possible strategies and/or activities which might be used to achieve the outcome(s). While the strategies and/or suggestions presented are not intended to be rigidly applied, they will help to further clarify the specific curriculum outcome(s) and to illustrate ways to work toward the outcome(s) while maintaining an emphasis on problem solving, communications, reasoning, and connections. To readily distinguish between activities and instructional strategies, activities are introduced in this column of the layout by the symbol ❐. As well, curriculum extensions intended for students in the Level 1 course are indicated with the *** ** *   * ** *** symbol. This symbol not only brackets text discussing differentiation for students in the Level 1 course, but also appears at the top of each page on which such text is located.

The third column of the two-page spread, "Worthwhile Tasks for Instruction and/or Assessment," might be used for assessment purposes or serve to further clarify the specific curriculum outcome(s). As well, those tasks regularly incorporate one or more of the four unifying ideas of the curriculum. These sample tasks are intended as examples only, and teachers will want to tailor them to meet the needs and interests of the students in their classrooms. The final column of each display is entitled "Suggested Resources" and will, over time, become a collection of useful references to resources which are particularly valuable with respect to achieving the outcome(s).

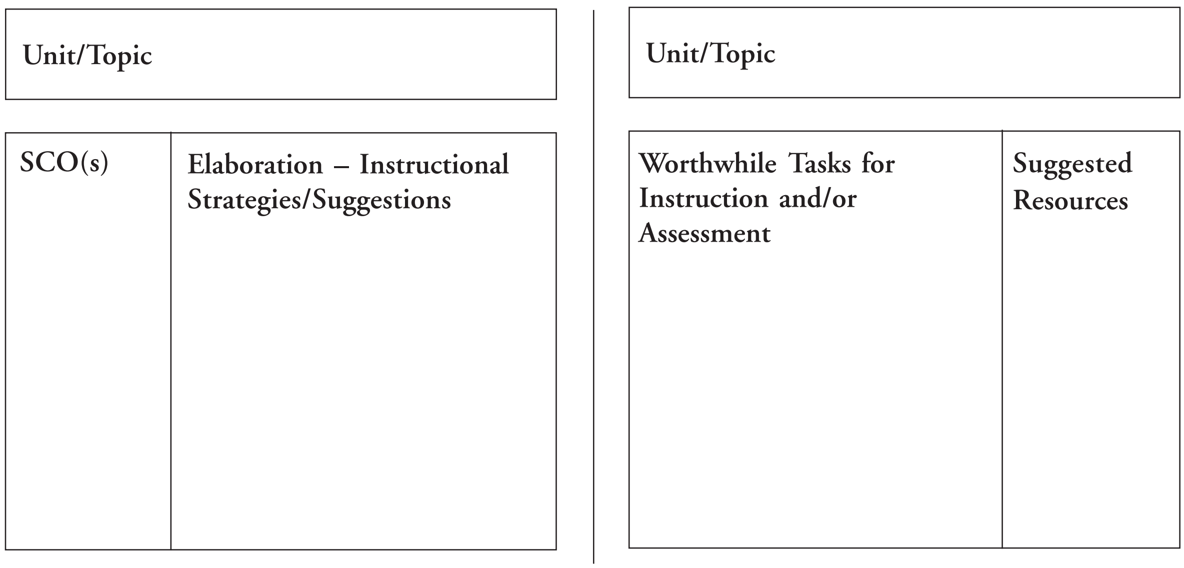


Figure 4: Layout of a 2-Page Spread

Specific Curriculum Outcomes
(by GCO)

GCO A: Students will demonstrate number sense and apply number theory concepts.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

ii) *order real numbers, represent them in multiple ways and apply appropriate representations to solve problems*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

A6 **develop an understanding of factorial notation and apply it to calculating permutations and combinations**

Elaboration

A6 Students will find it valuable to develop factorial notation in connection with tree diagrams and the fundamental counting principle, and will understand *n!* to represent the number of possible arrangements of *n* distinct objects. Factorial notation will be applied when calculating permutations and combinations, in connection with SCOs G7, G8 and B8. Unit 3, pp. 98, 100

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

ii) *derive, analyze and apply computational procedures in situations involving all representations of real numbers*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

B8 **determine probabilities using permutations and combinations**

Elaboration

B8 Evaluating permutations and combinations is addressed in SCOs G7 and G8. Students apply these counting techniques to the determination of probabilities in B8. Unit 3, p. 100a

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iii) *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

C20 (111) **represent circles using parametric equations**

C20(111) This outcome is intended for students in the Level 1 course only. Students will use a system of two parametric equations to represent a circle and, thereby, reach a greater understanding of the means by which technological devices "draw" circles. This outcome will be developed in connection with SCOs C36(111) and C37(111). Unit 4, p. 134

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v) *analyze and explain the behaviours, transformations and general properties of types of equations and relations*

C36 (111) **demonstrate an understanding of the relationship between angle rotation and the coordinates of a rotating point**

C36(111) This outcome is intended for students in the Level 1 course only. Students will need to extend their previous understanding of trigonometric ratios involving acute angles in right triangles to the broader range of values associated with the 360-degree rotation of a point. This understanding will be fundamental to their success with SCOs C20(111) and C37(111). Unit 4, p. 134

C37 (111) **describe and apply parameter changes within parametric equations of circles**

C37(111) This outcome is intended for students in the Level 1 course only. Students will do this in connection with, but following, SCOs C36(111) and C20(111). Unit 4, p. 136

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iii) *apply measurement formulas and procedures in a wide variety of contexts*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

D1 develop and apply formulas for distance and midpoint

Elaboration

D1 Within the context of coordinate geometry, students will develop formulas for distance and midpoint. They will apply these formulas in analytical proofs (see SCOs E11 and E4) and when developing and applying the equation of a circle (E15).
Unit 4, pp. 122, 124, 130

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *iii) analyze and apply Euclidean transformations, including representing and applying translations as vectors* SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to **E3 write the equations of circles and ellipses in transformational form and as mapping rules to visualize and sketch graphs**	E3 Writing equations of circles and ellipses in transformational form and as mapping rules facilitates the identification of key characteristics which describe their shapes and simplify sketching. Students will need to master the technique of completing the square to rewrite equations from general form (i.e., $Ax^2 + By^2 + Cx + Dy + E = 0$) into transformational form (or standard form). (Note: Facility with completing the square will serve students well in work with quadratics in later courses.) Students will develop these skills in conjunction with work on SCOs E13, E14, E15 and E16. Unit 4, p. 132
iv) represent problem situations with geometric models and apply properties of figures **E4 apply properties of circles**	E4 Students will not only learn numerous properties of circles, but will also apply them to practical situations and to situations involving both Euclidean and analytical proofs. (See, for example, related SCOs D1, E11 and E15.) Unit 4, pp. 112, 114, 118, 128, 130
v) make and test conjectures about, and deduce properties of and relationships between, 2- and 3-D figures in multiple contexts **E5 apply inductive reasoning to make conjectures in geometric situations**	E5 Mathematical reasoning is one of the unifying ideas in mathematics teaching and learning. Here, students will explore geometric situations (e.g., through paper folding) and make conjectures regarding geometric properties which they observe. This inductive process will precede proving and applying theorems deductively. Unit 4, pp. 112, 114, 122

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

	Elaboration
E7 investigate and make and prove conjectures associated with chord properties of circles	**E7** Students will make conjectures regarding the properties of chords in circles, prove them, and apply them in both Euclidean and analytical situations. (See also SCOs E5, E11 and E15.) Also, students will examine the concept of converse (E12) within the context of chord properties. Unit 4, pp. 112, 114, 116, 120, 132
E8 investigate and make and prove conjectures associated with angle relationships in circles	**E8** Angles within circles (e.g., central angles, inscribed angles) and their relationships provide a context for students to build inductive and deductive reasoning skills. Unit 4, pp. 112, 116, 126
E9 investigate and make and prove conjectures associated with tangent properties of circles	**E9** Tangent properties of circles provide a context for students to build inductive and deductive reasoning skills. Unit 4, pp. 112, 126
vi) demonstrate an understanding of the operation of axiomatic systems and the connections among reasoning, justification and proof	
E11 write proofs using various axiomatic systems and assess the validity of deductive arguments	**E11** Mathematical reasoning is one of the unifying ideas in mathematics teaching and learning. Students will be exposed to Euclidean, transformational and analytical systems of proof, and will write and assess both synthetic (e.g., Euclidean) and analytical (i.e., coordinate) proofs. Unit 4, pp. 112, 114, 118, 120, 122, 124, 126, 132
E12 demonstrate an understanding of the concept of converse	**E12** Students will explore the concept of converse in the context provided by SCO E7. Students should understand that only some theorems have converses which are true. Also, students should understand the use of "if and only if" terminology in situations in which a theorem and its converse are both true. Unit 4, pp. 112, 114, 116

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

Elaboration

viii) explore and apply, using multiple representations, circles, ellipses and parabolas and, in 3-D, spheres and ellipsoids

E13 analyze and translate between symbolic, graphic, and written representations of circles and ellipses

E13 Students will translate between various representations of circles and ellipses as needed to best explore their characteristics and solve problems. Facility with these translations will be developed in connection with SCOs E3, E14, E15 and E16. Unit 4, p. 132

E14 translate between different forms of equations of circles and ellipses

E14 Students will translate among transformational, standard and general forms of circles and ellipses. Developing these skills will be closely associated with SCO E3. Unit 4, p. 132

E15 solve problems involving the equations and characteristics of circles and ellipses

E15 Students will use the knowledge and skills developed around SCOs E3, E13, E14 and E16 to solve problems (both contextual and analytical) involving circles and ellipses. Unit 4, pp. 128, 130, 132

E16 demonstrate an understanding of the transformational relationship between a circle and an ellipse

E16 Students will visualize an ellipse as a circle stretched either horizontally or vertically. Mastery of the algebraic representation of stretches will be built upon students' experiences with stretches in quadratic contexts in Year 10. E16 will be addressed in conjunction with E3, E13, E14 and E15. Unit 4, p. 132

GCO F: Students will solve problems involving the collection, display and analysis of data.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *i) understand sampling issues and their role with respect to statistical claims* SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to	
F1 draw inferences about a population from a sample	**F1** Since it is often impractical to gather information about entire populations, sampling is a common statistical technique. Students will need to understand issues with respect to sampling strategies and sample size in order to properly draw inferences from sample data. These issues are addressed in SCOs F2 and F4. Unit 1, pp. 34, 38
F2 identify bias in data collection, interpretation and presentation	**F2** Students should understand that some sampling methods (e.g., convenience, self-selected) may produce samples that are not representative of the population as a whole. They will need to identify the bias that can enter the interpretation of results based on such sampling. Unit 1, p. 34
F4 demonstrate an understanding of how the size of a sample affects the variation in sample results	**F4** By conducting experiments/simulations and examining the data collected, students should understand that larger sample sizes increase the likelihood that the statistical results will approximate expected values or population characteristics. This outcome will be addressed in connection with SCO F15. Unit 1, pp. 38, 40, 42, 44
iv) determine, interpret and apply as appropriate a wide variety of statistical measures and distributions	
F7 draw inferences from graphs, tables and reports	**F7** Students will draw inferences based on such information as data distribution (e.g., normal vs. skewed) and data variability (e.g., range). This outcome will be addressed in connection with SCOs F10 and F15. Unit 1, pp. 36, 40, 42, 44
F8 apply characteristics of normal distributions	**F8** Students will connect previous learnings regarding normal distributions and standard deviation to 95% confidence intervals as determined by formula. Unit 1, p. 56

GCO F: Students will solve problems involving the collection, display and analysis of data.

	Elaboration
F9 construct, interpret and apply 90% box plots	**F9** (Note: This outcome falls within the optional portion of the statistics unit and will only be addressed if the optional portion is included in students' studies.) Students will extend their previous knowledge of (50%) box plots to the construction, interpretation and application of 90% box plots. This will occur in conjunction with SCOs $G1_2$ and $G3_2$. Unit 1, pp. 46, 48, 50, 52
F10 interpret and apply histograms and probability bar graphs	**F10** In connection with SCOs $G3_2$ and F7, students will construct and interpret histograms and frequency and probability bar graphs, and recognize conditions under which they begin to resemble normal distributions. Unit 1, pp. 40, 42, 44
F11 determine, interpret and apply confidence intervals	**F11** Students will determine confidence intervals by formula, and interpret and apply the results. (Note: Optionally, students may also determine confidence intervals using 90% box plots.) In conjunction with SCO F8, confidence intervals will also be connected to normal distributions and standard deviation. Unit 1, pp. 52, 54, 56, 58, 60

v) design and conduct relevant statistical experiments and analyze and communicate the results using a range of statistical arguments

F14 (111) **formulate hypotheses and null hypotheses**	**F14(111)** This outcome is intended for students in the Level 1 course only. In connection with SCO F18 (111), students will formulate and test hypotheses. The chi-square statistic (see F19(111) and F20(111)) will be introduced as one means of hypothesis testing. Unit 1, p. 58
F15 design and conduct experiments/surveys to explore sampling variability	**F15** By designing and conducting experiments/simulations and examining the data collected, students will understand the impact of sampling issues on statistical results. This outcome will be addressed in connection with SCO F4. Unit 1, pp. 38, 40, 42, 44
F16 demonstrate an understanding that the type of experiment/ survey affects the organization and communication of results	**F16** Students will understand that the type of questions asked, and the type of sampling conducted, will influence the presentation and interpretation of results. Unit 1, p. 36

GCO F: Students will solve problems involving the collection, display and analysis of data.

vi) test hypotheses using appropriate statistics

F18 (111) test hypotheses and interpret the results

F19 (111) apply and interpret the chi-square (X^2) statistic

F20 (111) collect data about two populations and analyze it using the chi-square statistic

Elaboration

F18(111) This outcome is intended for students in the Level 1 course only, and will be addressed in connection with SCOs F11 and F19(111). Unit 1, pp. 58, 60

F19(111) This outcome is intended for students in the Level 1 course only. Students will apply the chi-square statistic to test hypotheses. This outcome will be addressed in conjunction with SCOs F20(111) and $G2_2$(111). Unit 1, pp. 60, 62, 64, 66

F20(111) This outcome is intended for students in the Level 1 course only. Students will use the chi-square statistic to test a null hypothesis with respect to two populations, in addition to comparing a population to a theoretical model. Unit 1, pp. 68, 70

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *design and conduct experiments a/o simulations to model and solve a wide variety of relevant probability problems, and interpret and judge the probabilistic arguments of others*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

$G1_3$ develop and apply simulations to solve problems

$G1_3$ In general, students will design simulations to model problem situations and use them to solve the problems. Further, students in the Level 1 course will use simulations to model binomial trials. (See also SCO G11(111).) Unit 3, pp. 94, 104

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ii) *build and apply formal concepts and techniques of theoretical probability*

$G2_3$ demonstrate an understanding that determining probability requires the quantifying of outcomes

$G2_3$ Students will understand that determining probabilities requires counting both the number of possible outcomes and the number of successful outcomes. In this context, tree diagrams, area models, the fundamental counting principle, and permutations and combinations are all means of counting. Unit 3, p. 84

$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events

$G3_3$ Students will understand that, if an event A can occur in n ways and an unrelated event B in m ways, then A and then B can occur in n x m ways. They will also understand how to apply the principle when events are related to (dependent upon) one another. Unit 3, pp. 84, 86

G4 apply area diagrams and tree diagrams to interpret and determine probabilities of independent and dependent events

G4 Students will use diagrams to assist with quantifying outcomes and determining probabilities. Unit 3, p. 88

GCO G: Students will represent and solve problems involving uncertainty.

	Elaboration
G5 (111) **determine conditional probability**	**G5(111)** This outcome is intended for students in the Level 1 course only. Students will determine conditional probabilities both by counting techniques and by formula. Unit 3, pp. 90, 92
G7 **distinguish between situations that involve combinations and permutations**	**G7** Students will distinguish between situations involving unordered collections of objects (i.e., combinations) and those involving an ordering of objects (i.e., permutations). Making these distinctions will be critical with respect to calculating permutations and combinations (SCO G8) and determining probabilities involving permutations and combinations (B8). Unit 3, pp. 96, 100
G8 **develop and apply formulae to evaluate permutations and combinations**	**G8** By exploring situations involving permutations and combinations, students will develop and apply formulae to calculate them and determine probabilities involving them. This outcome will be addressed in conjunction with SCOs A6, G7 and B8. Unit 3, pp. 90, 100, 100a
iv) relate probability and statistical situations	
G9 **demonstrate an understanding of binomial expansion and its connection to combinations**	**G9** Students will understand and apply the pattern of exponents in the binomial expansion, as well as the connection between the coefficients and combinations. This outcome will be addressed in connection with SCO G10. Unit 3, p. 102
G10 **connect Pascal's Triangle with combinatorial coefficients**	**G10** Students will generate Pascal's triangle and connect its entries with combinatorial coefficients. Unit 3, p. 102
G11 (111) **connect binomial expansions, combinations, and the probability of binomial trials**	**G11(111)** This outcome is intended for students in the Level 1 course only. Repeated trials of experiments with only two possible outcomes are called binomial trials. Students will connect combinations and probability with respect to binomial trials. This outcome will be addressed in conjunction with SCOs $G1_3$ and G12(111). Unit 3, pp. 104, 108
G12 (111) **demonstrate an understanding of and solve problems using random variables and binomial distributions**	**G12** Students will understand random variables and solve problems involving binomial distributions. Unit 3, pp. 106, 108

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration

$G1_2$ construct and apply 90% box plots and normal probability distributions, and determine confidence intervals

$G1_2$ (Note: Part of this outcome falls within the optional portion of the statistics unit and will only be addressed if the optional portion is included in students' studies.) In connection with SCO F11, students will interpret normal probability distributions in terms of confidence intervals. Constructing and applying 90% box plots falls within the optional portion of the unit. Unit 1, pp. 46, 50, 52, 56

$G2_2$ (111) connect probability with the chi-square (X^2) statistic to interpret its meaning

$G2_2$(111) This outcome is intended for students in the Level 1 course only, and will be addressed in conjunction with SCO F19(111). Unit 1, pp. 64, 66

$G3_2$ graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability

$G3_2$ (Note: Part of this outcome falls within the optional portion of the statistics unit and will only be addressed if the optional portion is included in students' studies.) In connection with SCOs F7 and F10, students will graph and interpret frequency and probability bar graphs. Interpreting 90% box plots falls within the optional portion of the unit. Unit 1, pp. 42, 44, 46, 50, 52

Independent Study

Elaboration

Further to the topics that involve SCOs which contribute to students achieving the general curriculum outcomes, an Independent Study unit is included in *Geometry and Applications in Mathematics 111/112*. This unit is intended to assist in the development of students' independent learning skills while allowing them to explore topics outside the prescribed curriculum. While the following SCOs fall outside the framework of GCOs A through G, they are to be considered integral to the curriculum.

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I1 Students will research mathematical topics outside the prescribed curriculum and demonstrate an understanding of their topics by sharing the results of their research with their peers. (See also SCO I2.) Unit 2, pp. 74, 76, 78, 80

I2 communicate the results of the independent research

I2 Students will present the results of their research to their peers in one of a number of possible formats. Note that the presentation of results will be intimately connected to the achievement of SCO I3. Unit 2, pp. 74, 76, 78, 80

I3 demonstrate an understanding of the mathematical topics presented by other students

I3 Students will provide information in one or more of a number of formats to demonstrate that they have learned from the presentations of their peers. This outcome will be achieved in connection with SCO I2. Unit 2, pp. 78, 80

Unit 1
Statistics

(15-20 Hours)

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F1 draw inferences about a population from a sample

F2 identify bias in data collection, interpretation, and presentation

Elaboration – Instructional Strategies/Suggestions

F1 A television poll is conducted to find the proportion of the public that watches a particular television program on a particular night and if it is watched by more women than men. A quality engineer must estimate what percentage of bottles rolling off an assembly line are defective. In both these situations, information is gathered about a large group of people or things. The expense of contacting every person or inspecting every bottle, not to mention the time it would take, is formidable. So information is gathered about only part of the group (a sample) in order to draw conclusions about the whole group (the population). Choosing a representative sample from a large and varied population can be a complex task. It is important to be clear about what is or who is the population to be described, and exactly what is to be measured.

Some possible areas of concern—

- How can a sample be chosen so that it is truly representative of the population?
- If a sample from one population differs from a sample from a different population, can you infer that the two populations also differ?
- Does the size of the sample make a difference?

F2 To sample people at a mall is fast, cheap, and convenient. Think about which people in the mall might be invited to participate. Often it is those who are well dressed, respectable looking, and friendly, because they look easier to approach. Often the sample from malls will over-represent the middle class and retired, and under-represent the poor. Convenience samples often produce unrepresentative data. When an error occurs due to bad sampling, the difference between the results obtained and the truth about the whole population is called bias.

F1/F2 Students need some experience looking at surveys and/or polls and the results obtained. They need to read reports describing these results to get a sense of all the variables that are in play that may have an effect on the accuracy and reliability of the results.

Students should be asked to identify other sampling methods (convenience, call-in surveys, voluntary response) that might result in bias, and explain why. Students should discuss the characteristics of various types of sampling, stratified, systematic, self-selected (eg., voluntary response), random, convenience, and cluster. For example, in voluntary response sampling, people choose to respond or not. In convenience sampling, the interviewer makes the choice. What is needed to eliminate bias is an impersonal way to choose the sample. Random sampling allows every individual or object an equal chance to be selected.

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F1

Activity

1) Mr. Smith came into class with a box that he said contained 400 popsicle sticks. On some he had put a red dot, on some a blue dot, and on the rest a green dot.
   a) Mr. Smith asked the students, “How can you determine if there are more sticks marked with red dots than with blue or green dots?” What do you think?
   b) Mr. Smith shook the box vigorously, put his hand in and mixed the sticks even more. He then asked students to tally the results as he walked around the classroom asking different students to pull out 3 sticks and say what colour the dots were. The first student pulled out three sticks and said “2 blue and 1 red.” Do you think there are more blue sticks in the box than green? How confident do you feel with your answer? Explain.
   c) After 10 students had selected, the tally looked like this: blue–12, red–8, green–10.
      Mr. Smith asked the following questions for students to answer:
      i) After 30 sticks have been selected, which colour stick do you think occurs the most in this box? Explain.
      ii) Describe the population and the sample.
      iii) Sally thinks that there are the same number of each colour in the box. Why do you think she says this?
      iv) How many trials should we look at before we can feel confident? Explain.
   d) Mr. Smith gathered another 30 stick colours. The results of all 60 now looked like this: blue–22, red–13, green–25.
      i) Has your prediction changed based on these new results? Explain why or why not.
      ii) What are you most confident about predicting at this time?
   e) After 90 selections, the results were blue–30, red–20, green–40
   f) Mr. Smith asked students to predict which colour appeared more in the box and to use words like population, sample, sample size, confidence, proportion or percentage.

F1/F2

Activity

2) The Mathematics Association of Atlantic Canada (MAAC) wanted to survey their membership. The question was “Would you be willing to increase your annual dues from $15.00 to $100.00 in order to support the implementation of the new math program?” The survey was conducted during a meeting of the New Brunswick Math Teachers Association whose members belong to the larger association, the MAAC. One hundred and twenty-seven members reported; seventy-two said yes. On tha basis of these results, the president of the MAAC felt he had to support and raise the dues of all members of the MAAC to $100.00 per year. Ask students to respond to the following: Write a report to the president pretending you are a math teacher from Newfoundland and Labrador who belongs to the MAAC. Inform the president about your feelings about how he has conducted business. Make reference to possible bias, and the sampling technique used by the Association. In your report make suggestions about how you think the survey should have been conducted, and explain your reasons. Make reference to the kind of sampling you would use and how you would make it effective.

Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W. H. Freeman and Company, NY

Landwehr, James M. et. al. “Exploring Surveys and Information from Samples,” *Quantitative Literacy Series*, Dale Seymour Publications, 1987

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F7 draw inferences from graphs, tables, and reports

F16 demonstrate an understanding that the type of experiment/ survey affects the organization and communication of results

Elaboration – Instructional Strategies/Suggestions

F7 Data, or numerical statistics derived from data, are essential for making many kinds of decisions in our lives. These data are useful, however, only if we can organize and present them so that they can be more easily interpreted. A few numbers computed from the data—averages, percents and the like—can be very helpful. Some numbers computed from the data lead to statistical inference (e.g., mean, standard deviation), e.g., drawing conclusions and stating our confidence about the conclusions.

F16 Different kinds of data may result from different kinds of surveys. For example, numerical data might result from questions like "How many hours", "How much time", or "How much money?" or from gathering measured data in an experiment. To interpret these results, means, medians, or modes may be used. Some surveys, often called polls, ask respondents to answer "yes" or "no" or only allow one of two possible responses. Results from these might be recorded as the "total number of yeses ..." or as a proportion or percentage. Other questionnaires may lead to extended answers where categorizing may be the way to report results.

F7/F16 Students should investigate graphs, tables, and displays of samples of data obtained in experiments and surveys. Students might come up with their own topics for a survey and survey the students in their class. Students should look at one-another's displays as a whole class, and discuss the results that can be interpreted from them. They should be encouraged to explain what they know based on the information on the display. From the display, they should be able to draw some conclusions about and describe the population that the sample represents, and attempt to make conclusions based on the sample.

For example, Caleb might ask every third person who comes through the classroom door if he/she has completed his/her math homework. If 9 of the 11 people said yes, Caleb would calculate the percentage of "yes" responses (about 82%), display the results in some creative way, and hypothesize that about 80% of students in this classroom (population) do their math homework.

To test the hypothesis, Caleb might perform repeated samples over the next few days. The class might notice that different samples from the same population differ. From the results of samples, he might be able to make predictions and state how confident he might be about his predictions.

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F7/F16

Performance

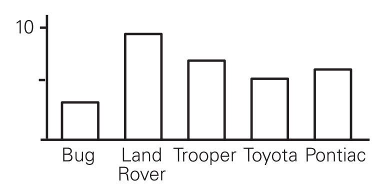
1) Anne wants to conduct a survey within her school. She would like to randomly select 100 students from within the student population. She asks the principal for a list of names for the entire student body.
   a) What strategy might Anne have in mind for selecting the 100 students?
   b) Mrs. Jones, the principal, suggests that she just go to the first three classrooms and conduct her survey with the students within those rooms. Comment on Mrs. Jones' approach. Will this satisfy Anne? Explain.

2) A magazine for health foods and organic healing wants to establish that taking large doses of vitamins will improve health. The editors ask readers who have regularly taken vitamins to write to the magazine describing their experiences. 2574 readers reply. How do you think the results will be reported?

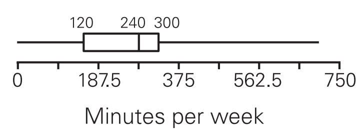
3) Gotham City Newspaper is seeking the opinion of the people of the city. An advertisement asked, "Should handgun control be tougher? You call the shots. Phone to vote yes: (900) 555-1621, or to vote no: (900) 555-1620. There is a 50¢ charge." How do you think the results will be reported?

4) Based on a sampling of 20 students at a mall, Jeffery told his dad that the average allowance among his friends was $10.00/week.
   a) How do you think he arrived at that conclusion?
   b) Comment on any possible bias.
   c) Describe the sample and the population.

5) Bob asked his classmates which kind of vehicle was their favourite. The graph displays the results. Give as much information as you can about the results of this survey.



6) Colin conducted a survey about homework and made a 50% box plot with the results. What do you think he should conclude? Be as thorough as possible.



Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W. H. Freeman and Company, NY

Landwehr, James M. et. al. "Exploring Surveys and Information from Samples," *Quantitative Literacy Series*, Dale Seymour Publications, 1987

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F15 design and conduct experiments/surveys to explore sampling variability

F4 demonstrate an understanding of how the size of a sample affects the variation in sample results

F1 draw inferences about a population from a sample

Elaboration – Instructional Strategies/Suggestions

F15 A sampling is conducted to get information about a population, and if chosen at random, the sampling is expected to represent the population. Your class may decide to ask a sample of your school population whether they watch a particular television program. Gallup may decide to do the same survey sampling from the population of Atlantic Canada. The results of these samples from different populations are likely to vary. In fact, if repeated samples are taken from the same population, the results are expected to vary from sample to sample. So, can results from samples be trusted—and how is trust affected by sample size? To explore this, students need to look more closely at sampling variability. Students should use containers from which to draw samples, coin flipping, die rolling, card flipping, and random number generators to gather data, and perform simulations with technology.

F15/F4 Students can begin to explore variability by designing and conducting simple surveys and/or experiments:

❒ Students should fill a container with objects that are all the same (e.g., M & Ms). They should designate one colour to be the "yeses" and take a sample of ten, recording how many yeses. From this, they should estimate the percentage of yeses in the container. They might give an interval that is as small as possible, but that they think contains the population percentage. Have them determine what the actual count is, and if it fits into their estimated interval. Have them repeat the whole procedure and determine if the estimate is more accurate as the sample size increases.

When students perform experiments like the above they will see that individual sample results vary, but that, as they repeat the experiment over and over and accumulate the results, they may be able to predict the actual result with more certainty. Students should be able to conclude that they are more likely to get an exact match between an observed result and what was expected (the expected is based on common sense, or experimental probabilities) when the sample size is small, but are more likely to get an approximate match when the sample size is large.

F15 Students will be particularly focussed on sampling issues, comparing samples from the same populations and comparing samples from different populations to enable them to interpret meaning and communicate it to others. In the sampling process, students will design and conduct statistical experiments/surveys that will provide a context and the sample data from which the variability can be explored.

F1 Students have learned that sampling a population to determine an outcome is cheaper than asking every member of the population their answer. However, the problem for students is how they relate the information from the sample to the population. It should be clear that there is a large number of possible samples which may be selected from a population.

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F15/F4/F1/A3

Performance

1) The school board conducted a census (a survey that involves the entire population) at the local high school. On the basis of the results, they said that because 40% of the student body responded in favour, a school day would run from 10am to 4pm. Gizelle and Mariette could not believe that 40% of the students responded favourably. They decided to conduct their own survey.
   a) Design the question they should ask to get a yes/no answer.
   b) Explain how they might conduct their study.
   c) Here are the results of what they actually did. They each asked 10 students a question which when answered yes showed that they were in favour of the time change. They repeated that 5 times each with a different 10 students each time.
   6 5 7 6 8 4 7 6 8 5 Display and interpret the data.
   d) Their teacher reminded them about sample size. They conducted the survey, again asking 50 students, each 10 times. Here are the results:
   29 28 35 30 29 27 31 30 26 27
   e) They were very sure now that the school board's results were wrong. Explain why.
   f) Before they approached the school board, they wanted to be sure. They obtained help from other students. Each of 100 students was to ask 100 students in the school the same question. The results are shown to the right. Explain how these students support Gizelle and Mariette. How should they communicate their results to the board to show the board that they must be wrong?

# Yeses	Frequency
52	1
53	4
54	0
55	8
56	7
57	8
58	7
59	10
60	9
61	10
62	5
63	8
64	8
65	5
66	1
67	4
68	4
...	0
74	1

2) Using the three sets of data above, explain why Terry says that he is more likely to get the expected value (40%) with a small sample size, but that he is more likely to almost get the expected value with a large sample size.

Suggested Resources

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F15 **design and conduct experiments/surveys to explore sampling variability**

F4 **demonstrate an understanding of how the size of a sample affects the variation in sample results**

F7 **draw inferences from graphs, tables, and reports**

F10 **interpret and apply histograms and probability bar graphs**

Elaboration – Instructional Strategies/Suggestions

F15/F4 Students might begin their exploration of variability by simulating a survey in which one out of every two (50%) respondents is expected to respond "yes" to a question.

Suppose it is known from a census taken by the community school association that exactly 50% of the students at Yore High School have a TV in their bedroom. You want students in your class to sample the student population at Yore School to determine how likely it is that they will obtain a sample proportion close to 50%. You might have them do this by asking random samples of 10 or 20 students during recess or noon hour. Or, you could have them simulate the sampling process, using another method, like tossing a fair coin, since a fair coin has an expected value of 50% (e.g., heads turn up half the time). You want the students to see how variability occurs in small sample sizes, but also how it improves when the number of trials increases and as sample size increases. So, ask students to toss 10 coins (sample size) ten times (number of trials) and record, each time, the number of heads that turn up. Students should be able to explain that one trial constitutes asking 10 students if they have a TV in their bedroom, and recording the result. This would be repeated 9 more times to get 10 trials. To increase the number of trials, have students pool their results. If you have a class of 30, and students are working in pairs, pooling the data gives you 150 trials, each of sample size 10.

F7/F10 Have students construct a frequency bar graph of the results. Have students examine the bar graph for variability. What is the range of the data? Is the distribution beginning to take on a shape of a normal curve? Students should calculate the mean for their own ten trials and compare it to the mean for the class for 150 trials. The bar graph and results should be recorded and displayed in the classroom for reference later.

F15/F4/F7/F10 The entire simulation discussed above should be repeated, but using a larger sample size, like 50. However, since students have already performed this using actual coins, this would be a good time to simulate the whole experiment using technology. If students are using graphing calculators, they can simulate the tossing of 50 coins, 10 times by using the command randBin (50, 0.5, 10). The 0.5 is the theoretical probability of tossing a head with a fair coin. Students should discuss why this command will perform that simulation. The results can be stored into a list, and the list can be used to produce a frequency bar graph. The list can then be analysed for the mean value and standard deviation. Instead of pooling the class data, each student can simulate 150 trials and produce their own bar graphs, means, and standard deviations.

F10 Students should be able to see from their comparison of frequency bar graphs that as the sample size increases and the trials increase, the range of values in the bar graph decreases, and the values cluster more around the mean (taller and skinnier bars), and the standard deviation decreases.

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F15/F4/F1/F7/F10

Project

1) a) There is probably some issue or concern where you might be interested in knowing how others feel. You are to conduct a yes/no survey on a topic where people are likely to have an opinion. Your teacher should approve your topic. You need to decide upon
   - i) who is the population
   - ii) how you will choose your sample, sample size
   - iii) how the people being sampled will feel about their response being anonymous

   b) Students will turn in a write-up on their project. Included in the write-up should be:
   - i) what the question is
   - ii) how and why the population was selected
   - iii) how you selected the people to sample
   - iv) how you ensured that people would respond honestly
   - v) sample size, and why you determined that size
   - vi) the results
   - vii) the conclusions based on the results

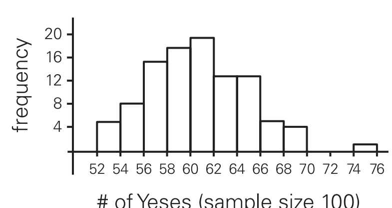
   Be prepared to give a 5 minute presentation to the class when you should discuss the process followed, interesting issues that developed, and why you stated the conclusion you did.

F10

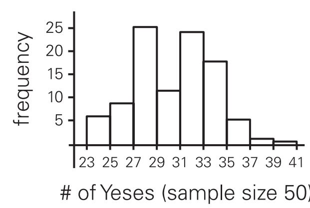
Performance

2) Consider the three plots A, B, and C. Each of A and B represent 100 trials, but A has a sample size100, while B is 50. C has a sample size 50, but only 10 trials. Examine and discuss these for variability. Discuss
   a) range
   b) distribution shape
   c) clustering about mean
   d) effect of sample size
   e) dispersion of data
3) Construct a 50% box plot for each and compare the box plots.

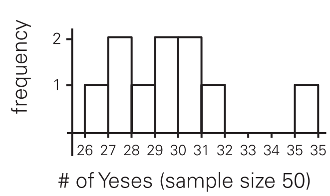
A



B



C



Suggested Resources

Garfunkel, Salomon, *Consortium for Mathematics and its Applications* (COMAP), *For All Practical Purposes*, W.H. Freeman & Co., NY

Addenda: *Data Analyses and Statistics, grades 9-12*, Burrill, Gail et al., NCTM, 1992

Landwehr, Swift, Watkins, Exploring Surveys and Information from Samples, *Quantitative Literacy Series*, Dale Seymour Publications, 1987

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F4 demonstrate an understanding of how the size of a sample affects the variation in sample results

F15 design and conduct experiments/surveys to explore sampling variability

F7 draw inferences from graphs, tables, and reports

F10 interpret and apply histograms and probability bar graphs

$G3_2$ graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability

Elaboration – Instructional Strategies/Suggestions

F4/F15 The results obtained in drawing samples from populations, when graphed, form a sampling distribution. Students need to examine distributions of sample results in order to better understand the behaviours of populations. One way to do this is to have students examine a sampling distribution from an experiment in which outcomes are equally likely. For example, if students are flipping 4 coins, how likely is it that they would get two heads and two tails? This is how the class could simulate the probability that 2 of the 4 children in a family are girls.

- Have students toss 4 coins and record the number of heads that turn up. This will tell them the number of girls in the family. Have students repeat this 40 times (or pool class data to get 40 trials) and complete a chart like the

heads	sample proportion	tally	frequency	proportion of all trials
0	0/4 = 0.00	III	3	3/4 = 0.075
1	1/4 = 0.25	~~IIII~~ III	8	8/40 = 0.200
2	2/4 = 0.50	~~IIII~~ ~~IIII~~ ~~IIII~~ I	16	16/40 = 0.400
3	3/4 = 0.75	~~IIII~~ ~~IIII~~	10	10/40 = 0.250
4	2/4 = 1.00	III	3	3/40 = 0.075

one following:

Note that the outcomes for numbers of heads on the four coins are not equally likely. Students should be able to determine why.

F7/F10/$G3_2$ Students should make a frequency bar graph of the results, showing frequency versus number of heads. Students should then make a probability bar graph by replacing the frequencies with the experimental probabilities (proportion of all trials column in the table) on the vertical axis. They should then answer questions that deal with the probability of events occurring. For example, have them answer "Did all of the numbers of heads have the same experimental probability? Is the probability of having only one girl greater than the probability of having only one boy? What is the probability of having at least 2 girls?"

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F15

Activity

We want to find the probability that a family with 6 children has 4 girls. We will use coins to represent children and heads to represent girls.

a) What other random device might you use to simulate the situation?
b) Toss the 6 coins all at once, and record the number of heads in a tally chart like the one below.

heads	sample proportion	tally	frequency	proportion of all trials
0				
1				
2	$\frac{2}{6} \doteq 0.33$			
3				
4				
5				
6				
Total			50	

c) After one toss what would be the sample proportion of girls?
d) Will you get this same sample proportion each time you toss the six coins?
e) Now toss the 6 coins, all at once, 50 times or pool the class results until you have 50 tallies. Complete the table above.

F7

f) What percentage of the time will there be 4 girls? Are you surprised? Explain.
g) Do this experiment again, only this time use "randBin (6, .5, 50)" on your calculator. Compare the results.
h) Construct a tree diagram of the situation to see the theoretical probability. How close were your simulations to the theoretical?
i) Complete this sentence. If we observe 6 randomly chosen students, the probability of this group containing 1, 2, or 3 girls is ______.

Suggested Resources

Statistics—Sampling

Outcomes

SCO: In this course, students will be expected to

F4 **demonstrate an understanding of how the size of a sample affects the variation in sample results**

F15 design and conduct experiments/surveys to explore sampling variability

F7 draw inferences from graphs, tables, and reports

F10 interpret and apply histograms and probability bar graphs

$G3_2$ **graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability**

Elaboration – Instructional Strategies/Suggestions

F4/F15/F7/F10/$G3_2$ Students should learn that an experiment like the one they just performed is called a binomial experiment because there are only two possible outcomes—heads or tails. Using technology, they can perform many similar experiments. For example, on the TI-83, using randBin (4, 0.5, 100) they could produce 100 trials, or using randbin (8, 0.5, 50), they could toss 8 coins, 50 times. Histograms on the TI can be replaced with probability bar graphs by changing the Freq: under the Xlist: choice to a list where the experimental probabilities are stored.

Students should compare the experimental probabilities with the theoretical probabilities by making a chart or tree diagram to show that each toss of 4 coins results in a possible 16 outcomes (not 4)—HHHH, HHHT, HHTH, HTHH, ... This will help develop an understanding of "equally likely."

Note that there is	1 outcome that results in 0 heads
	4 outcomes that result in 1 head
	6 outcomes that result in 2 heads
	4 outcomes that result in 3 heads
and only	1 outcome that results in 1 head

With technology like the TI-83 graphing calculators, students can determine theoretical probabilities by using binompdf (4, 0.5, {0, 1, 2, 3, 4}). This will show the theoretical probabilities of tossing 4 coins for 0 heads, 1 head, ..., 4 heads. The "pdf" stands for "probability density function."

With the understanding of the above concepts and procedures, students should be able to answer questions such as:

- Are you more likely to get a sample proportion of exactly 0.5 if you toss 4 or 8 coins? (4 coins)
- Are you more likely to get a sample proportion of between 0.25 and 0.75 if you toss 4 or 8 coins? (8 coins)
- Are you more likely to get exactly 10 heads from tossing 20 coins or exactly 50 heads from tossing 100 coins? (10 from 20)
- Are you more likely to get a sample proportion of between 0.25 and 0.75 from tossing 20 or 100 coins? (100)

Note that there is a connection between this work and work with binomial probability that is presented in Unit 3. Teachers should make every effort to make the connection obvious and clear.

Statistics–Sampling

Worthwhile Tasks for Instruction and/or Assessment

F4/F15/F7/F10

Performance

1) Perform the experiment shown on p. 43 again, for a sample size 12. Use technology as in step (g): randBin (12, .5, 50). Complete a table as previously shown. Store the data in a list and construct a frequency bar graph with 0 to 12 on the horizontal axis.
   a) What is the most likely number of heads? [Ans: 6 or 7]
   b) Estimate the probability of getting 5, 6, 7, or 8 heads.
   c) Compare this table with the previous table.
      i) Are you more likely to get a sample proportion of exactly 0.5 heads if you toss 6 coins or 12 coins?
      ii) Are you more likely to get a sample proportion of heads between 0.25 and 0.75 if you toss 6 coins or 12 coins?

F10/$G3_2$

Performance

2) Revise your frequency bar graph to a probability bar graph. Explain how you would answer question 1b) above, using the probability bar graph.
   a) Use the probability bar graph to determine the estimated probability that a family with 12 children would have only 1 girl (heads), 8 girls, all girls.

Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W.H. Freeman & Co., NY

Statistics—90% Box Plots/Confidence Intervals (Optional Curriculum)

Outcomes

SCO: In this course, students will be expected to

F9 **construct, interpret, and apply 90% box plots**

$G3_2$ **graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability**

$G1_2$ **construct and apply 90% box plots and normal probability distributions, and determine confidence intervals**

Elaboration – Instructional Strategies/Suggestions

The curriculum (involving 90% box plots) that begins here and continues until early on p. 54 is optional. A decision regarding its inclusion will depend upon a number of factors, including time contraints. (Note: Time should be a less significant factor with respect to the Level 1 course.)

F9/$G3_2$/$G1_2$ Making box plot summaries sometimes makes it easier to understand sampling distribution. In this section students will construct 90% box plots. The box contains at least 90% of the sample proportions. By stacking the box plots (as in p. 50) for population percentages from 5% to 95% at 5% intervals, students can construct an intuitively simple, yet statistically accurate framework for understanding the idea of confidence interval. The application of the 90% box plot is accurate and will help students to understand reports such as:

> For this sample size, the reported figure of 58% who favour eating peanuts at ball games is accurate to 3 percentage points.

Students will learn that the confidence interval for the percentage favouring peanuts at ball games is $58\% \pm 3\%$, or 55%-61%. If random samples are repeatedly taken, and the confidence interval from each sample is computed, about 9 out of 10 such confidence intervals will contain the actual population percentage (58%).

Statistics–90% Box Plots/Confidence Intervals (Optional Curriculum)

Worthwhile Tasks for Instruction and/or Assessment

F9/G3$_2$/G1$_2$

Pencil and Paper

1) The data to the right were obtained by sampling in a population where it was known that 50% of the population would respond "yes" to a question concerning the ownership of a computer. 100 random samples of size 20 were taken. You can simulate this with randBin (20, .5, 100).

   a) When you make a 90% box plot from 100 trials, how many of the sample data should ideally be in the box?
   b) How many of the sample data should each whisker ideally contain?
   c) Add a column to the table and complete it for sample proportion.
   d) What is the sixth smallest data sample value from the 100 trials? What does it represent? What sample proportion corresponds to it?
   e) What is the sixth largest data sample value from the 100 trials? What does it represent? What sample proportion corresponds to it?
   f) Make the 90% box plot from the 100 trials.
   g) How many sample data values actually ended up in the box?
   h) To construct a 90% box plot from 200 random samples of size 20, how many sample data values must you count in from either one to determine the edges of the box?

# Yeses	Frequency
0	0
1	0
2	0
3	0
4	1
5	2
6	5
7	12
8	11
9	10
10	16
11	21
12	8
13	8
14	4
15	1
16	1
17	0
18	0
19	0
20	0

...continued on p. 49

Suggested Resources

Statistics—90% Box Plots/Confidence Intervals (Optional Curriculum)

Outcomes

SCO: In this course, students will be expected to

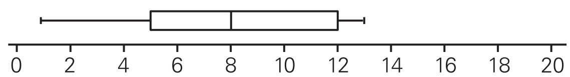
F9 construct, interpret, and apply 90% box plots

Elaboration – Instructional Strategies/Suggestions

F9 Using their technology (randBin (20, .4, 50) students can randomly select 50 data values from a known population of 40% sample size 20. They can organize the data to help them construct a 90% box plot. Below are the data:

# of yeses	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...20
Frequency	0	1	0	0	0	3	5	11	7	8	4	5	3	3	0...	...0

Since the 90% box plot should contain the middle 90% of the data, students should use the lower 5% to form the left whisker and the upper 5% to form the right whisker. The remaining values go in the box.



The box contains 5 yeses to 12 yeses or a sample proportion range from 0.25 (5 yeses out of 20) to 0.6 (12 yeses out of 20). On the bais of this sample, a student might say that if he or she were to repeat this procedure with a sample size 20, a student would get between 5 and 12 yeses 90% of the time. Typically, between 25% and 60% of the respondents would answer “yes”, 9 times out of 10.

Students should be able to explain why a median value of 8 is expected. They might respond that since 20 people were asked the question, and the population was known to respond yes 40% of the time, then 8 out of 20, which is 40%, might be the expected mean or median value.

Statistics–90% Box Plots/Confidence Intervals (Optional Curriculum)

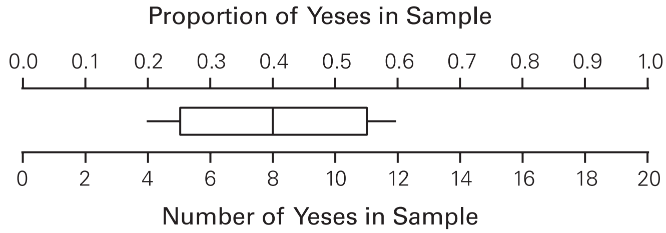
Worthwhile Tasks for Instruction and/or Assessment

... continued from p. 47

F9/G3_2/G1_2

Pencil and Paper

2)



a) It is known that 47% of voters voted to lift a ban on walking the downtown streets of Anytown after 10:30 pm. If you take a random sample of 20 voters, is getting 9 (sample proportion of 0.45) who voted this way a likely or unlikely sample proportion? How do you know?

b) If you take a random sample of size 20 from a population with 40% yeses, will each sample proportion below be a likely or unlikely sample proportion? Explain your response for each.

i) 0.40 ii) 0.65 iii) 0.20 iv) 0.90 v) 0.35

3) Forty percent of all plain M&Ms are brown. If you take a random sample of 20 M&Ms, tell whether each number of brown M&Ms below is likely or unlikely.
a) 9 brown b) 2 brown
c) 15 brown d) 7 brown

Journal

4) Complete this sentence:
If a random sample of size 20 is taken from a population with 40% yeses, 90% of the time, students will get a sample proportion of yeses between _______ and _______.

5) According to a campus newsletter, a survey found that 40% of the voters in Atlantic Canada favour a longer school year (10 months). Suppose that you select a random sample of 20 Atlantic Canadians and learn that 4 favour a longer school year. If the survey is right, is 4 out of 20 a likely or unlikely sample proportion? Given this sample proportion, would you think the survey is right?

Suggested Resources

Statistics—90% Box Plots/Confidence Intervals (Optional Curriculum)

Outcomes

SCO: In this course, students will be expected to

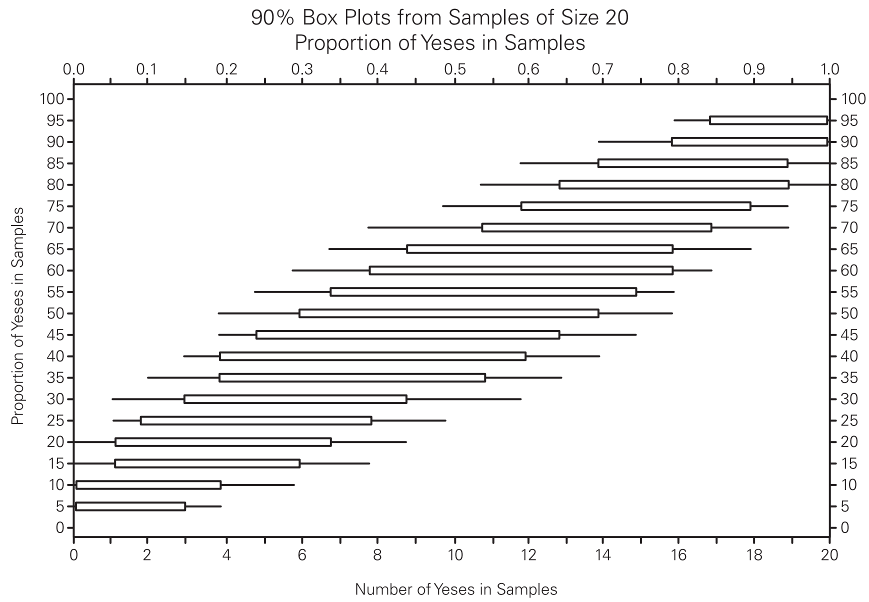
G1$_2$ construct and apply 90% box plots and normal probability distributions, and determine confidence intervals

G3$_2$ graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability

F9 construct, interpret, and apply 90% box plots

Elaboration – Instructional Strategies/Suggestions

G1$_2$/G3$_2$/F9 Give students a table of stacked 90% box plots, like that below, all from sample size 20. Each box plot represents a different known population percentage ranging from 0% to 100% in increments of 5%, displayed as a scale on the left-hand side of the table. At the top of the table is the sample proportion scale and at the bottom is the number of yeses scale. Have students use the table to interpret missing information.



There are different ways the table can be used:

One way would be, if the students know that 35% of the population indicated a particular response, students should be able to find a likely sample proportion. They would start in the left column to find the population percentage 35%, slide to the right until the box begins and read the sample proportion 0.20 (scale at the top), then to the end of the box. They would read 0.55 from the top scale, and interpret what that tells them: From a sampling of 20 people from a population that you know, 35% will answer yes, it's quite likely that 20% to 55% of the people (4 to 11 people) will answer yes. How likely is it? Since it is a 90% box plot, we can be 90% confident.

Another way would be, if they knew the interval of the number of people in a sample size 20 who said yes to some question, they could estimate with 90% confidence what the population percentage might be. If they knew that 6 people out of 20 said yes, they could estimate then that between 15% and 50% of the population from which the sample was taken would say yes.

continued ...

Statistics–90% Box Plots/Confidence Intervals (Optional Curriculum)

Worthwhile Tasks for Instruction and/or Assessment

$G1_2/G3_2/F9$

Pencil and Paper

1) Using a table of stacked 90% box plots of sample size 20 like that shown on the previous page, ask students to respond to the following:
   a) Looking at the box plot that represents a population in which 80% of the people would respond 'yes' to a question, what percentage of all data would you expect inside the box? Explain.
   b) If a random sample of 20 adults from this population were asked the same question, state whether the following results are likely or not, and explain.

      i) 20 yeses ii) 2 nos iii) 15 nos iv) 14 yeses

2) Suppose about 80% of adults support the right of school administrators to open school lockers or examine personal property for drugs, liquor, or other contraband. Is it likely or unlikely that a poll of 20 randomly selected adults would show
   a) just 2 favouring this practice?
   b) all 20 favouring it?
   c) 17 favouring it?
   d) 16 favouring it?

3) According to a census of inmates of juvenile correctional facilities, 80% of those under correctional supervision were male. If we take a random sample of 20 such inmates, is it likely or unlikely that 15 will be male?

4) Complete this sentence:
   If we draw a random sample of size 20 from a population with 80% yeses, we estimate that the proportion of yeses in our sample will be from _______ to ________ at least 90% of the time.

5) A teacher thought that 60% of the students in his school would have seen the movie ET, but when he asked 20 students at random, he learned that only 8 had seen this movie. Do you think he was wrong? Why or why not?

6) About 45% of all mathematicians in the United States are women. If we take a random sample of 20 mathematicians, are the following results likely or unlikely?

   a) 10 women and 10 men b) 15 women, 5 men c) 8 women, 12 men

7) Imagine you are taking a true/false test about some topic you didn't study about. For each of the 20 questions you discretely flip a coin and answer true if the coin lands heads and false if it lands tails. Are each of the following results likely or unlikely?
   a) a 100% score on the test
   b) a 90% score on the test
   c) an 80% score on the test
   d) a 70% score on the test
   e) What scores are you likely to get?

Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W.H. Freeman & Co., NY

Addenda: *Data Analyses and Statistics, grades 9-12*, Burrill, Gail et al., NCTM, 1992

Statistics—90% Box Plots/Confidence Intervals (Optional Curriculum)

Outcomes

SCO: In this course, students will be expected to

$G1_2$ **construct and apply 90% box plots and normal probability distributions, and determine confidence intervals**

$G3_2$ **graph sample distributions and interpret them using 90% box plots, probability bar graphs, and the language of probability**

F9 **construct, interpret, and apply 90% box plots**

F11 **determine, interpret, and apply confidence intervals**

Elaboration – Instructional Strategies/Suggestions

... continued

$G1_2$/$G3_2$/F9/F11 From another example we can state our confidence level. If 14 of 20 people surveyed answered yes to a question, by drawing a vertical line on our table from the number 14 on the lower scale, and observing which boxes were intersected, we would be 90% confident that between 50% and 80% of the population from which the sample was taken also say yes.

Students should now practise this with questions coming from various contexts, tables of varying sample sizes should be used.

Introduce the language of the "confidence interval." Students should be able to respond like ..."I took a random sample of 40 students at my school and asked them if they loved math. Because 13 of them said yes, I am fairly sure that if I ask another randomly selected 40 students at my school this question, between 25% and 45% will say yes. However, for every 100 times that I give such an interval, I expect to be right 90 times and wrong 10 times.

In the end, through practising getting confidence intervals from the tables of 90% box plots and varying sample sizes (40, 80, 100), students should be able to answer a question using language such as:

"From a sample size 20 in the grade 11 class, 14 students said yes to the question "do you brush your teeth at least twice a day?" From this we feel 90% confident that 50 to 85% of all students in grade 11 brush their teeth at least twice a day."

Students should notice that as the sample size increases the 90% box plots become proportionally shorter. For example, have students compare the 90% box plot for a known population of 40%, sample 20, with the corresponding box plot for sample size 40. They should note that the sample proportion has decreased from a range of 0.2 to 0.6 for a sample size 20 to 0.275 to 0.525 for a sample size 40.

Students should be able to explain why this is so, on the basis of their understanding developed earlier about outcome F4. These ideas continue the development of how confidence interval is related to sample size.

Statistics–90% Box Plots/Confidence Intervals (Optional Curriculum)

Worthwhile Tasks for Instruction and/or Assessment

$G1_2/G3_2/F9$

Pencil and Paper

1) Use a chart of stacked 90% box plots, sample size 20. A random sample of size 20 contains a sample proportion of 0.20 yeses. For which of the following population percentages is this a likely sample proportion?
   a) one with 5% yeses
   b) one with 10% yeses
   c) one with 15% yeses
   d) one with 20% yeses
   e) one with 25% yeses
   f) one with 30% yeses
   g) one with 35% yeses
   h) one with 40% yeses
   i) one with 45% yeses
   j) one with 50% yeses

2) a) A sample of 80 lower- and middle-class boys found that 8 had conduct disorders, such as stealing, fighting, and running away from home. For which population percentages is this a likely sample proportion?
   b) The same study also investigated about 80 lower - and middle - class boys with symptoms of hyperactivity. Of the boys in this sample, a proportion of 0.27 had conduct disorders. For which population percentages is this a likely sample proportion?
   c) Considering your answers to questions (a) and (b), do you think hyperactive boys are more likely than typical boys to have conduct disorders? Explain.

3) Compare the 90% box plots for random samples of size 80 with those for samples of size 20.
   a) Which chart has shorter box plots?
   b) Why do you think this chart has shorter box plots?

4) a) Suppose you take a random sample of size 20, and from that 6 people respond "yes" to the question "Do you love milk better than any other beverage?" From the chart of 90% box plots, sample size 20, which population percentages contain this sample proportion?
   b) Complete this statement: I am 90% confident that between __% and __% of the population from which I sampled would respond "yes."
   c) Complete this statement: If 14 people say "yes" to the question about milk from a sample of 20 people, I can be 90% confident that ...

5) Suppose you reach in a jar of marbles, pull out 40, and find that 18 are blue.
   a) Is this sample proportion likely if 60% of the marbles in the jar are blue?
   b) For which population percentages is a sample proportion of 18 out of 40 likely?

6) Atlantic Magazine gave each of 20 eighth graders three erasable pens and a nonerasable Stic ballpoint pen. The students used the pens for one week while doing their usual schoolwork. At the end of the week, 14 students preferred the Stic to any of the erasables. Assuming that the magazine selected the students and pens randomly, find the 90% confidence interval for the percentage of all eighth graders who prefer the Stic to these erasables.

7) Of the 20 students in question 6, 11 chose the Scripto as the best erasable pen.
   a) What is the 90% confidence interval for the percentage of all eighth graders who prefer the Scripto to the other erasables?
   b) Can you be fairly confident that at least half of all eighth graders prefer this erasable pen? Why or why not?

Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W.H. Freeman & Co., NY

Addenda: *Data Analyses and Statistics, grades 9-12*, Burrill, Gail et al., NCTM, 1992

Statistics—Confidence Intervals

Outcomes

SCO: In this course, students will be expected to

F11 determine, interpret, and apply confidence intervals

Elaboration – Instructional Strategies/Suggestions

F11 Students, at this point, should reflect on their understanding of confidence intervals. If students took random samples from a population and constructed a confidence interval for each sample, the population percentage should be inside 90% of the intervals constructed. Stated differently, students might say that in 10 out of every 100 surveys, the confidence interval will not contain the population percentage.

If polling organizations used 90% box plots, they would be wrong about 10 of every 100 surveys. They would rather not be wrong that often. Thus, students might think that they would more likely use 95% box plots so that they would be wrong only 5 times in every 100. Students should understand that a 95% box plot lengthens both the box and the confidence interval.

This marks the end of the section of optional curriculum dealing with 90% box plots. The prescribed curriculum for students in both Level 1 and Level 2 courses resumes here.

Students should be told that, in fact, polling organizations do not use box plots at all. Instead, they use a simple formula that determines the sampling error in a survey that was random sampling. Sampling error related to confidence interval. Here is how:

The formula for determining sampling error is $2\sqrt{\frac{p(1-p)}{n}}$, where p is the sample proportion and n is the sample size. The formula gives a decimal value which can be converted to a percentage. Using the formula requires that $np \geq 5$, and $n\,(1-p) \geq 5$. The formula $2\sqrt{\frac{p(1-p)}{n}}$ is based on the assumption that the sampling distribution is close to a normal distribution, a condition that is only true if $np \geq 5$ and $n\,(1-p) \geq 5$. The formula can be derived using a complicated statistical theory, and the theory says that when the formula is used, the population percentage (e.g., the mean response of the population) will be in the confidence interval at least 95% of the time. For example,

> Suppose in a sample of 100 people 68 say yes. Then n = 100, and $p = \frac{68}{100} = 0.68$; and np = (100)(.68) = 68, and $n(1–p)$ = 32, so, the sampling error is calculated using the above formula to be 0.0933 or about 9%. The 95% confidence interval for the true population percentage is 59% to 77% ($68\% \pm 9\%$). This means if a sample, chosen at random, results in 68% being in favour of some event, then one can predict with 95% confidence that, if another sample of the population was taken, between 59% and 77% would be in favour.

Students should practise using the formula a few times from varying contexts. These contexts can provide students the opportunity to interpret and apply confidence intervals.

Statistics–Confidence Intervals

Worthwhile Tasks for Instruction and/or Assessment

F11

Activity

1) Assume random sampling and ask students to find the sampling error as a percentage for each of the surveys that follow. Use the formula $\sqrt{\frac{p(1-p)}{n}}$.
   a) sample size: 25, number of yeses: 20
   b) sample size: 100, number of yeses: 80
   c) sample size: 400, number of yeses: 320
   d) sample size: 1600 number of yeses: 1280
   e) Looking at the sampling errors in the 4 surveys above, ask students to describe what happens when the sample size is multiplied by 4.
   f) See if the same pattern holds for these surveys whose sample sizes are multiplied by 9 each time.
      i) sample size: 20, number of yeses: 16
      ii) sample size: 180, number of yeses: 144
      iii) sample size: 1620, number of yeses: 1296
      iv) sample size: 14580, number of yeses: 11664
   g) Try it one more time for the following surveys:
      i) sample size: 24, number of yeses: 18
      ii) sample size: 120, number of yeses: 90
      iii) sample size: 600, number of yeses: 450
      iv) sample size: 3000, number of yeses: 2250
   h) What conclusion can you make? Explain.

Pencil/Paper

2. If the sample proportion is p = 0.5, what sample size gives a sampling error of
   a) 1% b) 3% c) 5% d) 8%?
3. In each of the surveys described below state a prediction from the sample results and qualify it with a confidence statement using 95% confidence:
   a) Of 127 cars travelling on a residential street with a new stop sign, only 71 came to a full stop.
   b) A survey of 200 mathematically gifted students found that 72% were near-sighted.
   c) In a survey of 350 full-time university students, 46% said that they drink to excess at least once a week.
4. An Atlantic poll with sample size 400 had these results: 128 support the Tories, 100 support Liberals, 152 were undecided. How would you report these results using confidence intervals to the public?

Suggested Resources

Garfunkel, Salomon, Consortium for Mathematics and its Applications (COMAP), *For All Practical Purposes*, W.H. Freeman & Co., NY

Statistics—Confidence Intervals

Outcomes

SCO: In this course, students will be expected to

F8 **apply characteristics of normal distributions**

$G1_2$ **construct and apply 90% box plots and normal probability distributions, and determine confidence intervals**

F11 **determine, interpret, and apply confidence intervals**

Elaboration – Instructional Strategies/Suggestions

F8 Students have been studying variability by exploring distributions of data as 90% box plots (optional). They have talked about being 90% confident and how the confidence interval is obtained from a chart. They have also talked about being 95% confident and how the confidence interval can be obtained from a formula. Students should try to connect what they have learned about confidence to what they learned about normal curves and standard deviation in Year 10.

F8/$G1_2$/F11 Review with students the important facts about normally distributed data.

- Approximately 68% of all results are within one standard deviation of the mean.
- Approximately 95% of all results are within two standard deviations of the mean.
- Approximately 99.7% of all results are within three standard deviations of the mean.

In a normal distribution, if students were to look at all the values that stretch from two standard deviations below the mean to two standard deviations above the mean, they should expect to find 95% of the values. This interval is a 95% confidence interval.

Students should know that the “strangeness” of a particular result can be determined by

- finding how many standard deviations the result is from the mean
- finding probability associated with this distance from the mean

For example, since 95% of the results lie within two standard deviations of the mean, the probability is 5% that a result differs by 2 or more standard deviations (assuming a normal distribution). It would be safe to conclude that any values outside that 95% around the mean would be considered quite “unusual” or “strange.”

Statistics–Confidence Intervals

Worthwhile Tasks for Instruction and/or Assessment

F8/G1$_2$/F11

Performance

1) A quality engineer must estimate what percentage of bottles rolling off an assembly line are defective. If the defective rate is 2% or greater, the engineer must shut down the production and improve the process.

   a) Suppose the defective rate is 2% and a random sample of 100 bottles is selected. Discuss the likelihood of the following:

   i) no bottles are defective

   ii) two bottles are defective

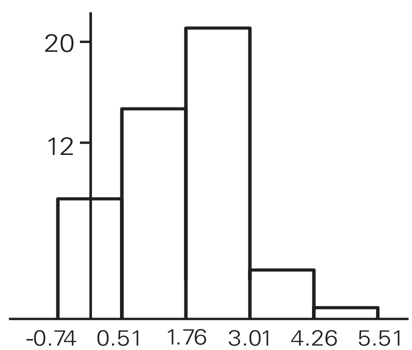
   iii) four bottles are defective

   iv) ten bottles are defective

   b) The following is a frequency bar graph sharing the results of 50 repetitions of the situation in (a) above. The mean was 1.76 bottles with a standard deviation of 1.25 bottles. State your confidence for the following and explain why.

   i) 3 bottles are defective

   ii) 6 bottles are defective



   c) Find a way to simulate selecting 1000 random samples of size 10 from a production line with a 4% defective rate. Construct a frequency bar graph. State with 95% confidence the number of defective bottles that would require the production line to be shut down for repairs. How would this number change if the engineer wanted to be 99% confident? Explain.

Suggested Resources

Statistics—Hypothesis Testing

Outcomes

SCO: In this course, students will be expected to

F14 (111) formulate hypotheses and null hypotheses

F18 (111) test hypotheses and interpret the results

F11 determine, interpret, and apply confidence intervals

Elaboration – Instructional Strategies/Suggestions

F14(111) In statistics, a hypothesis is a statement about something that is thought to be true. Often, hypotheses are formulated on the basis of the results of a survey or an experiment. For example, Felix might ask every third person who comes through the classroom door if he/she completed his/her math homework last night. Based on this survey, he might hypothesize that all students do math homework (since all students in his survey did their homework). The null hypothesis (negation of the hypothesis) would state that students do not complete math homework. Often a null hypothesis can be formulated without the need of a survey. For example, before flipping a coin, one would often hypothesize that 'the coin is fair.'

Research is often conducted to compare the mean results for two groups. One group is the "control" group and the other is the "test" group. Group members do not know which group they are in. Often a situation is being researched; for example, the effect a particular pill might have on some physical phenomena. The pill manufacturer hypothesizes that "the pill will have an effect." The research question is posed as the null hypothesis: "The pill will have no significant effect." The research is done to determine if there is reason to 'reject' the null hypothesis.

F14(111)/F18(111) Participants in both groups are given a pill - the control group takes a neutral pill (a placebo, not medical), while the test group is given the pill being tested. Over time data are collected and analysed in order to test the null hypothesis.

F18(111)/F11 Students can test a hypothesis or null hypothesis. For example, if the null hypothesis states that "this coin is fair" a student could flip it 100 times and expect to see 50 heads appear. If the coin landed with heads up 94 times, most people would think the coin is biased and reject the null hypothesis. If 51 heads appear, could the coin be called fair? Students will learn that the null hypothesis can be rejected only if the number of heads differs from the expected "significantly."

For example, suppose a teacher claims that exactly 10% of the people who have just left teaching say that students' lack of motivation was one of the main reasons they left. A study of 500 former teachers found that 8% gave this reason (*American Educator*, Summer 1986).

a) Construct the 95% confidence interval.
b) Should you tell the teachers that they are wrong?

Solution:

a) The sampling error is $2\sqrt{\frac{0.08(1-0.08)}{500}} = 2\%$. So the 95% confidence interval is 6% to 10%.

b) No. The 95% confidence interval includes the teachers' figure of 10%. Thus 10% could well be the exact percentage.

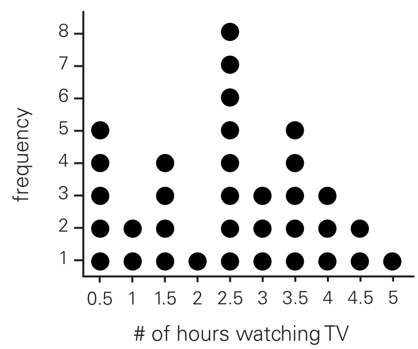
Statistics–Hypothesis Testing

Worthwhile Tasks for Instruction and/or Assessment

F14 (111)

Performance

1) Sharon put a bar graph up on the wall in her classroom and asked students to put a sticker in the bar appropriate to their behaviour last night.
   a) If possible, make a hypothesis for Sharon regarding the number of people in grade 11 who watch a certain amount of hours of TV in a night.
   b) If possible, make a hypothesis about which group, boys or girls, watches more TV in a night. If not possible, explain why not.
   c) Describe a survey from which Sharon might be able to hypothesize that girls or boys watch more TV at night.
   d) What null hypothesis might Sharon make about boys and girls and the amount of time each watches TV at night?



2) At noon time, Jeremy asked 50 students at random if they preferred math class or history class. 42 students said 'math'. What might Jeremy hypothesize? Explain. If he wanted to test this what might he do?

F18(111)/F11/F14(111)

Performance

3) Ronnie owns a Christmas tree lot. 1000 randomly selected trees were cut from this lot with a mean height of 2.0 m and a standard deviation of 4.5 cm.
   a) Describe how the trees might have been randomly selected.
   b) Ronnie makes a hypothesis about the trees on his lot. What might be his hypothesis?
   c) Assuming that the distribution of height is normal, describe how Ronnie might test his hypothesis, using normal curve and standard deviation.
   d) Marilyn bought 20 trees from a lot for her 20 customers. The mean height of her trees was 2.7 m. How confident are you that her trees came from Ronnie's lot? Explain.

Suggested Resources

Statistics—Chi-Square

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Outcomes

SCO: In this course, students will be expected to

F18 (111) **test hypotheses and interpret the results**

F11 **determine, interpret, and apply confidence intervals**

F19 (111) **apply and interpret the chi-square χ^2 statistic**

Elaboration – Instructional Strategies/Suggestions

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* **F18(111)/F11/F19(111)** Students have already learned that, if observed data is more than 2 standard deviations from the mean of a given normal distribution, it is unlikely that the observed data came from that distribution. Earlier in the elaboration of outcomes, students have used charts of 90% box plots and normal curves and standard deviations to compare a sample to a known population or standard behaviour. When the population or standard behaviour is not known, students can use the chi-square χ^2 test as another method for making a decision about an event that does not seem to conform to an expected behaviour. Students need a method to tell whether a particular χ^2 result justifies rejecting the null hypothesis, or a method to tell how unusual it is to get a particular χ^2 when the null hypothesis is true.

When a result does not conform to an expected behaviour and is considered strange, it means that it would be unusual to get a result this far or further from the mean. Students might be asked to consider how they might measure how strange the results might be. For example, which is more strange—14 heads in 20 flips, 54 heads in 100 flips, or 500 heads in 1000 flips? Some students may simply calculate the numerical differences, some the percentage differences. Statisticians have found that neither the numerical difference "observed–expected" nor the percentage differences $\frac{\text{observed - expected}}{\text{total number}}$ can be used to measure this strangeness when the sample sizes are different. However, if $\frac{\text{(observed - expected)}}{\text{total number}}$ is computed for each observed number, and the results are added, a more effective measure of "strangeness" will result. (This resulting number is called a chi-square statistic, χ^2 .) Students should note the resemblance between the above formula and that used for standard deviation.

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Statistics–Chi-Square

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Worthwhile Tasks for Instruction and/or Assessment

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* **F19(111)**

Performance

1) a) Describe a procedure you could use to test whether the 8-sided die Mr. Bloomingthal uses on his overhead is a fair die.
   b) Earlier in the week John's class performed an experiment to determine if a 6-sided die was fair or not. The class created a chart of 60 chi-square values. Could the chi-square statistic for Mr. Bloomingthal's die be compared to this chart from John's class? Explain.
   c) The χ^2 value for the 6-sided die in John's class was 7.0. The class agreed that this was not high enough to think this die was not fair. Would you expect the χ^2 value for Mr. Bloomingthal's die to be higher or lower than 7.0? Explain.

2) Ricky and his older brother Roddy often settle disputes by flipping coins. For example, they argue about who will select first when asked to pick a piece of cake from all those cut already. When the coin is flipped, Roddy calls heads and always seems to win. At least this is Ricky's claim. Ricky asked his teacher, Mr. Bloomingthal if they could test Roddy's coin for fairness. Explain what Mr. Bloomingthal should do to set up the distribution. Explain why he cannot use the distribution already established for 6-sided or 8-sided die.

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Suggested Resources

Statistics—Chi-Square

Outcomes

SCO: In this course, students will be expected to

F19 (111) **apply and interpret the chi-square (χ^2) statistic**

Elaboration – Instructional Strategies/Suggestions

F19(111) As an example of a χ^2 calculation discussed on pg 60, ask students to calculate χ^2 for a sample of 50 coin flips of which 20 were heads. The null hypothesis is that this coin is fair, so the expected number of heads is 25.

	heads	tails
50 coin flips	expected 25	expected 25
	observed 20	observed 30

$$
\begin{aligned}
\chi^2 &= \tfrac{5^2}{25} + \tfrac{(-5)^2}{25} \\
&= \tfrac{5^2}{25} + \tfrac{(-5)^2}{25} \\
&= 1 + 1 \\
&= 2
\end{aligned}
$$

By conducting many repetitions of this experiment with a fair coin and calculating the χ^2, students will see how the chi-square value is affected as the observed results get further and closer to the expected, (the χ^2 gets larger as the difference gets larger - or the stranger the result, the larger the χ^2). Students should conclude that

- for a fixed sample size, the bigger the numerical difference between the observed results and the expected results, then the bigger the χ^2 value.
- for a fixed numerical difference between observed and expected, the smaller the sample size, the bigger the χ^2.
- for a fixed percentage difference, the bigger the sample size, the bigger the χ^2.

All of this means, the “stranger” the results, the larger the χ^2 statistic. Or, the larger the χ^2, the more justified one would be in rejecting the null hypothesis. But students need a way of knowing how large the χ^2 value needs to be to be able to reject the null hypothesis.

Students should also realize that if the null hypothesis is rejected, it doesn’t necessarily follow that the opposite is true, or that there is a related specific alternative hypotheses. Similarly, if the null hypothesis is not rejected, it will be considered the likely option, but other options are possible.

Statistics–Chi-Square

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Worthwhile Tasks for Instruction and/or Assessment

*** ** * **F19(111)**

Pencil and Paper

1) Calculate the χ^2 value for each of the following situations.

   a) You obtain 508 heads after tossing a coin 1000 times.
   b) You obtain 28 heads after tossing a coin 40 times.
   c) Marla says that these two situations are equally likely to happen because they both result in 8 more heads than expected.
   d) Do the χ^2 values from (a) and (b) reflect Marla's thinking? Explain.
   e) Which situation do you think is more 'strange'? Explain.

2) Calculate the χ^2 values for each of the following situations.

   a) You obtain 15 threes when tossing a 6-sided die 25 times.
   b) You obtain 42 fives when tossing a 6-sided die 70 times.
   c) Pierre says these two situations are equally likely to happen. Why do you think he feels this way? Explain.
   d) Do the χ^2 values from (a) and (b) reflect Pierre's thinking? Explain.
   e) Which situation do you think is more 'strange'? Explain.

3) Richard and David are planning a math conference. They are led to believe that 72% of the membership of the Math Teachers Association (MTA) are expected to come to the conference. Richard sets the registration rates accordingly. David does not trust this projection and decides to sample the membership. In a mailout, 56 members of the 91 who replied said they would attend, 35 said no.

   a) What would David hypothesize from this? What would be his null hypothesis?
   b) Calculate the χ^2 statistic for David's data.
   c) Should Richard change the registration rates on the basis of the sample results? Explain. * ** ***

Suggested Resources

Statistics—Chi-Square

Outcomes

SCO: In this course, students will be expected to

F19 (111) **apply and interpret the chi-square (χ^2) statistic**

G2$_2$ (111) **connect probability with the chi-square (χ^2) statistic to interpret its meaning**

Elaboration – Instructional Strategies/Suggestions

F19(111)/G2$_2$(111) To determine how unusual it is to get a particular χ^2 value, students need a distribution of χ^2 values with which they can compare their observed χ^2 value. To establish this distribution or table, students could perform repeated experiments in situations where the null hypothesis is true, and, for each, calculate the χ^2 value. If they collect, say, 50 chi-square values and put them in a stem and leaf plot—like the one below—from lowest to highest to form a distribution, students can interpret this plot to see how likely it is a certain χ^2 value or higher might occur.

Let us say that students were each given a fair 6-sided die to throw 60 times. They were asked to record how many times each number from 1–6 on the die turned up. They were then asked to calculate the chi-square value for their data (remember this would have them sum six calculations of $\frac{(\text{observed - expected})^2}{\text{expected}}$).

χ^2	Results
0	244
1	0246666
2	00244446888
3	226
4	68
5	0266
6	0224468
→ 7	004 ← (Observed χ^2 = 7)
8	246688
9	8
10	2
11	
12	02

8|2=8.2

The stem and leaf plot to the left represents their chi-square calculations. It is a distribution of chi-square values for the results of tossing a fair die.

Students were then given a new die and asked to perform the experiment again, and calculate their chi-square value. Before they started they were told that some of them had a die that was loaded. They were asked to interpret their results, and, using the stem and leaf distribution, decide whether their die was fair or loaded. The null hypothesis was that their die was fair. After a while, Agnes called out that her chi-square value was 7.0, and she thought she might have a loaded die. The rest of the class was asked to help Agnes. They should examine the plot to determine the probability of an χ^2 value of size 7.0 or larger occurring. Thirteen of the 50 chi-square values are 7.0 or larger. Students would conclude that 7.0 or larger will occur about 13 out of every 50 times, or about 26% of the time. They should conclude that this does not seem unusual and they shouldn't reject the null-hypothesis. Agnes was disappointed.

Statistics–Chi-Square

Worthwhile Tasks for Instruction and/or Assessment

G2$_2$(111)/F19(111)

Performance

1) Given the stem and leaf plot that represents the χ^2 values for the number of heads recorded when flipping 100 coins 50 times, use it to determine the probability of obtaining the χ^2 value for Ricky's coin.

Stem	Leaf
0	00 00 04 04 04 04 04 04 04 04 04 04
.	16 16 16 16 16 16 16 16 16 36 36 36 36 36 64 64 64 64 64
1	00 00 00 00 00 44 44 44 44
.	96 96
2	56 56 56
.	
3	24
.	
4	00 00 00
.	
5	76
.	

a) On page 61, the χ^2 value for Ricky's coin was calculated. His coin had 59 heads in 100 tosses.

b) How many of the χ^2 values in the plot are the same or higher than Ricky's χ^2 value?

c) Calculate this probability $P(x^2 \geq 3.24) = ?$ and $2|56 \Rightarrow 2.56$ explain what it means.

d) Tony's coin turned up heads 52 times in 100 tosses. What was his χ^2 value? What is the probability of a χ^2 Tony's χ^2 value occurring? Would you say Tony's coin is fair or not? Explain.

e) Meredith's chi-square value was 2.41. Explain how to interpret her χ^2 value.

Suggested Resources

Statistics—Chi-Square

Outcomes

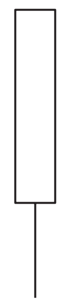
SCO: In this course, students will be expected to

F19 (111) **apply and interpret the chi-square (χ^2) statistic**

G2$_2$ (111) **connect probability with the chi-square (χ^2) statistic to interpret its meaning**

Elaboration – Instructional Strategies/Suggestions

F19(111)/G2$_2$(111) Students might want to display their chi-square distribution in a stem and leaf plot and construct, beside it, a one-tailed 90% box plot.



χ^2	Results
0	244
1	0246666
2	00244446888
3	226
4	68
5	0266
6	0224468
→ 7	004 ← (Observed χ^2 = 7)
8	246688
9	8
10	2
11	
12	02

8|2=8.2

The die throwing experiment can be analysed differently. The χ^2 statistic should be calculated for this experiment and compared to the values in the table.

The probability of getting an χ^2 statistic of 7.0 or larger from a situation where the null hypothesis is true can be determined. If the observed χ^2 value falls within that box, then that χ^2 value is likely to occur at least 90% of the time, and would not be thought of as unusual. When students use an established table of χ^2 values, a low probability from the table means that there isn't much probability that a result like this could occur by chance, and so students should reject the null hypothesis; e.g., they should conclude that the die is not fair (there is only a 10% probability of obtaining a chi-square value of 8.8 or higher).

The stem and leaf plot of chi-square values in the plot above represents a distribution of values that were obtained when there were 6 possible outcomes (six calculations of $\frac{(\text{observed - expected})^2}{\text{expected}}$ were added to obtain each chi-square value in the plot). This plot represents the distribution of χ^2 values that will be obtained every time a model with 6 outcomes is used. (A model that has 6 possible outcomes is said to model a situation with 5 degrees of freedom—a term that teachers should be aware of, but that is of less importance for students in this course.) If students wanted to use χ^2 distribution to judge the fairness of a coin, they would use a distribution that contained χ^2 values calculated by adding two calculations of $\frac{(\text{observed - expected})^2}{\text{expected}}$ – a distribution of χ^2 values representing one degree of freedom. Once such a distribution is established, it can be used to judge whether or not to reject the null hypothesis for any situation or model that has only two possible outcomes.

χ^2 value	corresponding probability
2.4	0.1213
2.6	0.1069
2.8	0.0943
3.0	0.0832
3.2	0.0736
3.4	0.0652

The die problem could be changed to have one degree of freedom by testing the number of threes to "not threes" or "other values." For example, when tossing a fair die 60 times, one would expect about 10 threes and 50 "not threes." If one observed 15 threes and 45 "not threes" then the χ^2 statistic would be calculated as the sum of two fractions: $\chi^2 = \frac{(15-10)^2}{10} + \frac{(45-50)^2}{50} = 3.0$. There is only an 8.3% chance of obtaining a χ^2 value of 3.0 or higher. Stated differently, a χ^2 value of 3.0 or higher will have a probability of occurring less than 8.3% of the time when the null hypothesis is true. Many students would use this result to reject the null hyptothesis—the die is not fair.

Statistics–Chi-Square

Worthwhile Tasks for Instruction and/or Assessment

F19(111)/G2(111)

Performance

# heads	Frequency
38	1
39	0
.. 42	... 0
43	1
44	1
45	4
46	3
47	4
48	5
49	3
50	2
51	7
52	4
53	1
54	2
55	1
56	3
57	1
58	3
59	1
60	3
61 ...	0 ...

1) a) In Ricky's class, Mr. Bloomingthal asked all the students to conduct the following experiment. He told them to flip a coin 100 times and record the number of heads. Each student was then asked to calculate the χ^2 value. Do this now for your coin.
   b) Calculate the χ^2 values for each of the results in the table from recording the number of heads in 100 tosses. How might you use technology to help?
   c) Organize the χ^2 values into an ordered stem and leaf plot in ascending order from top to bottom.
   d) Construct a one-tailed 90% box plot beside the stem and leaf plot.
   e) Interpret the values in the box with respect to the stem and leaf plot.
   f) Determine the χ^2 value for Ricky's coin. (59 heads)
   g) Do you think Ricky's coin is fair? Explain.

2) Mr. Bloomingthal wanted to do another experiment with his class, this time involving 6-sided dice. He gave each student a sheet of paper with the net of a 6-sided die drawn on the paper and asked the students to cut it out, fold it up, and tape the joints to form a cube. Three students were asked to tape a paper clip to the inside of one face before they taped the whole cube together. The cubes were then all mixed together.
   a) Describe how Mr. Bloomingthal would continue his experiment to find the three "loaded" dice.
   b) State the null hypothesis.
   c) Following are some of the χ^2 values obtained. Using the chart on p. 64, which results might cause rejection of the null hypothesis? Explain.

Mary 2.4	Richard 8.2	Jake 5.2	Zackary 2.7
David 6.7	Sharon 10.1	Tracie 8.1	Holly 6.9
Ron 4.8	Kate 7.1	Nate 13.4	Cindy 7.1
Ann 9.2	Belinda 4.3	Connor 3.6	Paul 8.2

   d) The following is the tally chart for Hilda's experiment. Calculate her χ^2 value, compare it to the chart, and make a conclusion, justifying your answer.
   e) Redo (d), but this time compare the number of times it comes up "not fours". Calculate the χ^2 value this time and use the chart on p. 64 to decide whether or not to reject the null hypothesis.

Suggested Resources

Statistics—Chi-Square

Outcomes

SCO: In this course, students will be expected to

F20 (111) **collect data about two populations and analyse it using the chi-square (χ^2) statistic**

Elaboration – Instructional Strategies/Suggestions

F20(111) Students should extend their understanding beyond comparing a population with a theoretical model (like coin tossing) to comparing two populations that they think may or may not differ in some respect. For example, a pill manufacturer has developed a pill that it thinks will have the same effect on men as on women. The two populations are the set of all men and the set of all women. Samples will be taken from each set to test the null hypothesis. The null hypothesis is that "the fraction of men who feel drowsy 30 seconds after taking the pill is the same as the fraction of women who feel drowsy 30 seconds after taking the pill."

The following activity introduces students to the process of finding the expected value for two-population cases, which is different than in population - model cases.

Complete the tables as if the null hypothesis stated above is true.

Situation 1:

- Assume 100 men and 100 women women
- In this group there are 120 drowsy people and 80 not drowsy

Situation 2:

- Assume 150 men and 50
- In this group there are 120 drowsy people and 80 not drowsy

	Women	Men	Total
Drowsy			120
Not drowsy			80
Total	100	100	

	Women	Men	Total
Drowsy	30	90	120
Not Drowsy	20	60	80
Total	50	150	

Students should find that if they think about what fraction of the total sample is drowsy, it will help them find the expected values. Since 120 of 200 are drowsy, then 60 men and 60 women in situation 1 are expected to be drowsy. In situation 2, 60% of 50 is 30, so 30 women and 90 men are expected to be drowsy if the null hypothesis is true. Knowing these values allows all four cells to be completed.

Statistics–Chi-Square

Worthwhile Tasks for Instruction and/or Assessment

F20(111)

Performance

1) Kenny had to do a survey for a math project. He wants to compare two populations—married and single people—with respect to their exercise habits. Respond to these different situations:
   a) Kenny's null hypothesis was that there was no difference. He interviewed 200 people, 100 married, 100 single. Set up a chart showing the expected values.
   b) Kenny was told that 35% of married people were light exercisers and 65% were heavy. Single people were 80% heavy exercisers. Set up a chart showing the expected value if he interviewed 120 married people and 60 single people.
   c) Given the following tables, complete them for expected values and describe the situation Kenny is dealing with.

i)

	Married	Single	Total
lots			70
little			130
Total	50	150	

ii)

	Married	Single	Total
lots			75
little			25
Total	75	25	

Suggested Resources

Statistics—Chi-Square

Outcomes

SCO: In this course, students will be expected to

F20 (111) collect data about two populations and analyse it using the chi-square (χ^2) statistic

Elaboration – Instructional Strategies/Suggestions

F20(111) Once students are fairly comfortable finding the expected values in situations like those described on the previous page, they should then be asked to complete the table as if the null hypothesis was not true. For example, the survey people have a feeling from earlier sampling that perhaps the pill affects men more than it affects women. Ask students what the table might look like if this were the case. Students should compare their observed data and agree that the number of men who said they were drowsy would increase, and so the number not drowsy would have to decrease.

	Married	Single	Total
Drowsy	20	100	120
Not Drowsy	30	50	80
Total	50	150	

A possible table of observed values that students might choose

Now students are ready to apply the χ^2 statistic to determine whether the two populations are really different in some respect. Students should follow the general procedure they have used for population—theoretical problems.

1) State the null hypothesis.
2) Determine the expected values for the null hypothesis.
3) Complete the χ^2 statistic.
4) Use a χ^2-probability chart to find the corresponding probability.
5) Decide whether or not to reject the null hypothesis.

There are four fractions to add instead of two (as with the population-theoretic cases). A possible calculation from the above problem might look like this:

$$\chi^2 = \frac{(20-30)^2}{30} + \frac{(100-90)^2}{90} + \frac{(30-20)^2}{20} + \frac{(50-60)^2}{60} = 13.3$$

As with the population—theoretical cases, the numerators of these fractions are all the same. Whenever the observed number in a cell is more than its expected number, the observed number in the other cell in that row or column must be an equal amount less than its expected number. This is why this situation is described as having "one degree of freedom."

Looking at a χ^2 probability chart, one can observe that a χ^2 value of 13.3 or higher is not even on the chart. The highest χ^2 value of 8.0 suggests that a χ^2 value higher than 8.0 will have a probability of occurring less than 0.5% of the time when the null hypothesis is true.

χ^2 statistic	corresponding probability
6.6	0.0102
6.8	0.0091
7.0	0.0082
7.2	0.0073
7.4	0.0065
7.6	0.0058
7.8	0.0052
8.0	0.005
one degree of freedom	

Statistics–Chi-Square

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Worthwhile Tasks for Instruction and/or Assessment

*** ** * **F20(111)**

Performance

1) a) Revisit #1, p. 69 and state Kenny's null hypothesis for (a).
   b) Copy the chart in (a) that you completed for the expected values and complete the chart, using the following information: During his interview, Kenny recorded 65 of the married people exercised lightly, the rest quite heavily. 80 of the single people exercised heavily, the rest very lightly.
   c) Calculate the chi-square statistic and interpret it. What conclusion would you state?

2) a) Revisit #1(b), p. 69. Based on the information, what hypothesis might Kenny want to test?
   b) Copy the chart of expected values for 1(b) and complete the chart with the following information. Of the 120 married people questioned, 50 were light exercisers, and the rest, heavy. The single people were 40 heavy, 20 light.
   c) Calculate the chi-square statistic and interpret it. What conclusion would you state?

Journal

3) Make up two questions which would require the use of a 2×2 table, like the tables used by Kenny above. For one question, make up a situation and data that demonstrate that the underlying population is significantly different. For the other, make up a different situation whose data indicate that any differences could easily be attributed to sample variation.

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Suggested Resources

UNIT 2:
INDEPENDENT STUDY

Unit 2
Independent Study
(10-15 Hours)

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

Elaboration – Instructional Strategies/Suggestions

I1/I2

The purpose of this independent study is 1) to prepare students for learning independently, and 2) to provide students with the opportunity to explore

- in more depth, mathematical content that they have been exposed to but would like to know more about
- new mathematical content areas not yet explored
- mathematical topics of interest
- historical studies and connections to the math we study
- mathematics in our lives, and related to careers
- mathematics through the Internet
- how people learn mathematics

Approximately 10–15 hours of class time should be devoted to this research project. Teachers should allow time for

- students to present the results of their research and learnings to other students (presentation time of about 10 minutes per student should be allowed for). If students are working collaboratively on this project, it is expected that each would be responsible for gathering certain information and thus could be held responsible for the oral presentation that deals with that part of the project.
- initial discussion and discovering of ways to get information, what it means to learn mathematics independently, and why that is important (the resources should supply activities to stimulate this).
- an introduction to topics not yet studied to whet students' appetites.
- discussing the expectations and assessment rubrics for the student presentations at the end of the unit and how they will be assessed during the unit.
- brainstorming, topic-webbing, developing action plans and time lines, conferencing ... .

Managing the Project

The managing of the project should be closely teacher-directed:

- Various topics will determine the appropriate group size.
- Some students may wish to work independently.
- Students will choose appropriate topics (perhaps from a teacher-prepared list) that are appropriate and of interest to them. Teachers should ensure the availability of reference resources (material and human) in or around the school, or the community and teachers should give final approval for each topic.

. . . continued

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

- *New Topics for Secondary School Mathematics*, *Matrices*, NCTM, 1988
- de Lange, Jan, *Meaningful Math, Matrices,* WINGS for Learning, 1992
- de Lange, Jan, *Flying Through Math*, T*rig, Vectors, and Flying*, WINGS for Learning, 1991
- Froelich, Gary et al., *Discrete Mathematics Through Applications*, W. H. Freeman and Company, New York, 1994
- Jacobs, Harold R., *Mathematics, A Human Endeavor*, Third Edition, W. H. Freeman and Company, New York, 1994
- Serra, Michael, *Discovering Geometry*, Second Edition, Key Curriculum Press, 1997
- Charles, Randall et al., *How to Evaluate Progress in Problem Solving*, NCTM, 1992

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

Elaboration – Instructional Strategies/Suggestions

. . . continued

I1/I2

- Brainstorming or topic-webbing should take place.
- Action plans should detail the tasks that have to be completed (it is assumed that each student will have responsibility for independent work within the structure of the project), for example:
  - write letters to gather data or request materials
  - make phone calls for information, read texts, newspapers, flyers, journals, and reports
  - complete library search
  - interview resource people
  - reflect on and share ideas with group members
  - prepare oral presentation
  - prepare written submission
- Each student or group should detail their own time lines to match the teacher's.
  - Deadlines should be determined for the work that has to be completed, and for bringing completed work to class. Regular conferences regarding progress are crucial. This includes conferencing with group members and with the teacher.

Suggested Topic and Content Areas

New Content

- vectors with respect to navigation and forces graph theory (4-colour problem and travelling salesman problem), or other topics found under the heading 'Discrete Mathematics' applications of matrices, such as Markov Chains and Leontief Input—Output models

Topics for More Depth

- proof in mathematics
- algebraic manipulation
- functions, compositions of functions—connection to art
- regression analysis
- the story on infinity and zero
- parametric equations
- conics from a geometric perspective

Mathematics in our Lives

- Fibonacci numbers—connecting to the world
- geometry in our lives—patterns, design, architecture
- mathematics in jobs—interview a person about how he/she uses math in his/her job
- do statistics lie?
- consumer mathematics
- career options
- leisure mathematics—non-routine, recreational problems, logic, math games, puzzles, games of chance
- the Internet as a source of mathematics information

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 **demonstrate an understanding of a mathematical topic through independent research**

I2 **communicate the results of the independent research**

I3 **demonstrate an understanding of the mathematical topics presented by other students**

Elaboration – Instructional Strategies/Suggestions

I1/I2/I3 Teachers might facilitate this unit by

- focussing 10–15 hours on the project all at one time-teachers should be aware of the time needed to gather and compile information
- spreading the project over a term, with some class periods being designated to the project introduction, and as checkpoints, each with a particular expectation, and for finalization, preparing and performing presentations
- integrating it with a topic going on at the same time in the classroom—statistics, algebra, indirect measurement

Expectations for assessment must be made clear to students:

- All students must be involved with the presentation on what mathematics they have researched:
  - oral presentation to class, or
  - oral presentation on video and played to the class, or
  - conversation between students or among group members in front of class or on video, or
  - teacher/student interview (private or in front of class) or,
  - some other variation.

Executive summaries must be distributed to the class at the time of presentation. This means that students should summarize the new mathematics learned so that other students can read over the summary, see a couple of examples, and have a pretty good feeling for the new topic.

Assessment

- Criteria for written submission should be made clear.
- Criteria for presentations should be made clear (students should not simply read their written submission).
- Criteria should be prepared by the teacher and discussed with the students prior to the assigning of the project.
- A rubric (students could help design it) should be included that allows for assessment on the written work as well as the presentation.
- Peer evaluation should occur during presentations and from each group member on the group effort.

continued ...

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

- *Assessment Alternatives in Mathematics*, Equals Publishing/Lawrence Hall of Science, University of California, 1989
- *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, NCTM, 1992
- Charles, Randall et al., *How to Evaluate Progress in Problem Solving*, NCTM, 1992

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

I3 demonstrate an understanding of the mathematical topics presented by other students

Elaboration – Instructional Strategies/Suggestions

. . . continued

I1/I2/I3 A possible rubric for the written component might look something like the following:

Top Level

- contains a complete report with clear, coherent, unambiguous, and elegant explanations
- includes clear and simple diagrams, charts, graphs, etc.
- communicates effectively to an identified audience
- shows understanding of the mathematical ideas and processes
- identifies all the important elements of the topic
- includes examples and counter-examples
- gives strong supporting arguments

Second Level

- contains good solid report with some of the characteristics above
- explains less elegantly, less completely
- does not go beyond the requirements of the project (or topic)

Third Level

- contains a complete report but the explanation may be muddled
- presents arguments but incomplete
- includes diagrams but inappropriate or unclear
- indicates understanding of mathematical ideas, but not expressed clearly

Fourth Level

- omits significant parts
- has major errors
- uses inappropriate strategies

See the books in the Suggested Resources column for more examples of rubrics for evaluating projects and open-ended activities.

I3 Students should collect executive summaries from students who are currently presenting. Students should ask questions for clarification at the end of presentations. Students might demonstrate their learnings from presentations of others by completing a questionnaire that focusses on the highlights of a presentation. On a test, teachers might ask students to discuss, showing examples, what was learned from any presentation. Another strategy for assessment might be through a conversation between the teacher and student about someone else's project.

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

UNIT 3:
PROBABILITY

Unit 3
Probability
(20 Hours)

Probability

Outcomes

SCO: In this course, students will be expected to

$G2_3$ demonstrate an understanding that determining probability requires the quantifying of outcomes

$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events

Elaboration—Instructional Strategies/Suggestions

$G2_3$ Every day students experience a variety of situations. Some involve making decisions based on their previous knowledge of similar situations.

- Should they do their math homework tonight or during their spare period before math class tomorrow?
- Should they challenge a friend to a game of racquetball or blockers?
- Should they buy a ticket on a car raffle?
- Should they take their umbrella today?

Before making the decision, what they must ask themselves is What is the chance of this decision working out in my favour?

In probability, the goal is to assign numbers between 0 and 1 inclusive to events that interest us, but for which we do not know the outcome.

In their previous studies (grades 7–9) students have created and solved problems using probabilities, including the use of tree and area diagrams and simulations. They have compared theoretical and experimental probabilities of both single and complementary events and dependent and independent events. They have examined how to calculate complementary events as well as two independent events, A and B, the probability of A and B is equal to $P(A) \times P(B)$.

Sometimes the task of listing and counting all the outcomes in a given situation is unrealistic, since the sample space may contain hundreds or thousands of outcomes.

$G2_3$/$G3_3$ The fundamental counting principle enables students to find the number of outcomes without listing and counting each one. For independent events, if the number of ways of choosing event A is $n(A)$ and the number of ways of choosing event B is $n(B)$, then $n(A \text{ and } B) = n(A) \times n(B)$, and
$n(A \text{ or } B) = n(A) + n(B)$.

The first is the multiplication principle, the second, the addition principle.

Sometimes events are not independent. For example, suppose a box contains three red marbles and two blue marbles, all the same size. A marble is drawn at random. The probability that it is red is $\frac{3}{5}$. If the marble is then replaced, the probability of picking a red marble again is $\frac{3}{5}$. However, if it is not replaced, then when another marble is picked the probability of its being red is now $\frac{2}{4}$. The second selection of a marble is dependent on the first selection not being returned to the box.

continued ...

Probability

Worthwhile Tasks for Instruction and/or Assessment

$G2_3/G3_3$

Activity

1) Two students are playing "grab" with a deck of special "grab" cards. One student has a triangular-shaped deck with 16 ones, 12 twos, 8 threes, and 4 fours. The other has a rectangular shaped deck with 10 each of ones, twos, threes, and fours. The decks are well shuffled and each student's plays the top card simultaneously. A "grab" is made when two cards match (a double).
   a) There are 40 cards in each deck. What is the total number of pairs of cards that could be played?
   b) How many of these are "double ones," that is, a one from the triangular deck and a one from the rectangular deck?
   c) How many are i) double twos?ii) double threes? iii) double fours?
   d) For equally likely outcomes, the probability of an event is "the number of outcomes that correspond to the event" divided by what?
   e) So, the probability of a double one is "what" divided by "the total number of pairs"?
   f) Use this principle and your answers to (c) to find the probability of i) a double one ii) a double two iii) a double.
   g) A circular deck has 10 ones, 20 twos, 10 threes, and no fours. Calculate the probability of a grab if a triangular deck is played against a circular deck.

Performance

2) Telephone numbers are often used as random number generators. Assume that a computer randomly generates the last digit of a telephone number. What is the probability that the number is
   a) an 8 or 9?
   b) odd or under 4?
   c) odd or greater than 2?

3) A airplane holds 176 passengers: 35 seats are reserved for business class, including 15 aisle seats; 40 of the remaining seats are aisle seats. If a passenger arrives late and is randomly assigned a seat, find the probability of that person getting an aisle seat or one in the business section.

4) Use the given table, which represents the number of people who died from accidents by age group to find the following. In each case assume that one person is selected at random from this group:
   a) the probability of selecting someone under 5 or over 74
   b) the probability of selecting someone between 16 and 64
   c) the probability of selecting someone under 45 or between 25 and 74

Age	Number
0-4	3,843
5-14	4,226
15-24	19,975
25-44	27,201
45-64	14,733
65-774	8,499
75 and over	16,800

Suggested Resources

Flewelling, Gary et at., *Mathematics 10 A Search for Meaning*. Toronto: Gage 1987.

Probability

Outcomes

SCO: In this course, students will be expected to

$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events

Elaboration—Instructional Strategies/Suggestions

continued ...

$G3_3$ How is the fundamental counting principle related to probability? Consider the marble situation described at the bottom page 84. The probability of selecting red is $\frac{3}{5}$, while the probability of selecting blue is $\frac{2}{4}$. The probability of selecting a red and a blue without replacement would be

P (r and b) = $\frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$.

Now, let us consider another situation:

Consider the experiment of a single toss of a standard die. There are six equally likely outcomes: 1, 2, 3, 4, 5, and 6. Define certain events as follows:

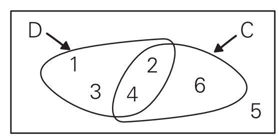
A: observe a 2
B: observe a 6
C: observe an even number
D: observe a number less than 5.

$P(A) = \frac{1}{6}$ (observe a 2), $P(B) = \frac{1}{6}$ (observe a 6). What about P(A or B)

(observe a 2 or 6)? This can be shown two ways: $\frac{n(A) + n(B)}{\text{total number of ways}} = \frac{1+1}{6} = \frac{2}{6}$

or P (A or B) = $P(A) + P(B) = \frac{1}{6} = \frac{1}{6} = \frac{2}{6}$. Will this be true for any two events?

The events "observe a 2", and "observe a 6" are called mutually exclusive events, or disjoint, because one can observe only a 2 or a 6, not both at the same time. On the other hand, events like C and D above have at least one element in common and therefore are not mutually exclusive.



Consider the events C and D. The event (C or D) includes all the outcomes in C or D or both.

That is, P(C or D) = P(observe an even number or a number less than five)

= P (observe 2, 4, 6, or observe 1, 2, 3, 4)

Every outcome except 5 is included in (C or D). Thus there are exactly five favourable outcomes. Thus P (C or D) $= \frac{5}{6}$

But $P(C) + P(D) = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$, which cannot be possible since it exceeds 1.

The outcomes 2 and 4 are contained in both C and D and must be removed. There is an overlap.

$$P(C \text{ or } D) = P(C) + P(D) - P(C \text{ and } D) = \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

$G3_3$

Performance

1) Discuss whether the following pairs of events are mutually exclusive and whether they are independent.
   a) The weather is fine; I walk to work.
   b) I cut a deck of cards and have a Queen; you cut a 5.
   c) I cut the deck and have a red card; you cut a card with an odd number.
   d) I select a voter who registered Liberal; you select a voter who is registered Tory.
   e) I found a value for x to be greater than –2; you found x to have a value greater than 3.
   f) I selected two cards from the deck, the first was a face-card, the second was red.
2) If 366 different possible birthdays are each written on a different slip of paper and put in a hat and mixed,
   a) find the probability of making one selection and getting a birthday in April or October
   b) find the probability of making one selection that is the first day of a month or a July date
3) A store owner has three student part-time employees who work independently of each other. The store cannot open if all three are absent at the same time.
   a) If each of them averages an absenteeism rate of 5%, find the probability that the store cannot open on a particular day.
   b) If the absenteeism rates are 2.5%, 3%, and 6% respectively for three different employees, find the probability that the store cannot open on a particular day.
   c) Should the owner be concerned about opening in either situation a) or b)? Explain.
4) There are 6 defective bolts in a bin of 80 bolts. The entire bin is approved for shipping if no defects show up when 3 are randomly selected.
   a) Find the probability of approval if the selected bolts are replaced, are not replaced.
   b) Compare the results. Which procedure is more likely to reveal a defective bolt? Which procedure do you think is better? Explain.
5) Mary randomly selects a card from an ordinary deck of 52 playing cards. What is the probability that Mary will select either an ace or a diamond? Below is Fred's solution. Explain what Fred is thinking. Will his attempt lead to a correct answer? Explain.

   $P(\text{ace or diamond}) = \frac{4+13}{52} = \frac{17}{52}$

Journal

6) Consider the table of experimental results. Comment on the following solution attempts.

	Seldane	Placebo	Control	Total
Drowsiness	70	54	113	237
No drowsiness	711	611	513	1835
	781	665	626	2072

   a) If one of the 2072 subjects is randomly selected, the probability of getting someone who took Seldane or a placebo is

$$\frac{781}{2072} + \frac{665}{2072} = \frac{1446}{4144} = 0.3489$$

   b) If one of the 2072 subjects is randomly selected, the probability of getting someone who took Seldane or experienced drowsiness can be found by:

$$\frac{781}{2072} + \frac{237}{2072} = \frac{1018}{4144} = 0.491$$

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

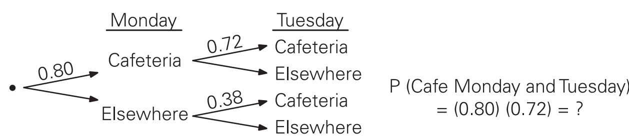
G4 apply area and tree diagrams to interpret and determine probabilities

Elaboration—Instructional Strategies/Suggestions

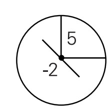
G4 Students have studied area and tree diagrams since grade 7 and have applied them to help establish the sample space, or the total number of possible outcomes in a situation. In this course their experiences with these diagrams will be extended to probability tree diagrams and area diagrams that will help students visualize and calculate the probabilities given certain situations.

Consider the following situation. Students at Yore High School have two choices for where to eat lunch, in the cafeteria or elsewhere outside the school. Mildred, the manager of the cafeteria, needs to be able to predict how many students can be expected to eat in the cafeteria over the long run. Mildred asks the math class to conduct a survey. The results show that if a student eats in the cafeteria on a given day, the probability that he or she will eat there the next day is 72%. If a student does not eat in the cafeteria on a given day, the probability that he or she will eat in the cafeteria the next day is 38%. On Monday, 80% of the students ate in the cafeteria. What can Mildred expect for Tuesday?

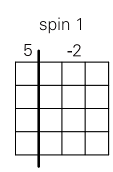
A good way to organize all these statistics is with a probability tree diagram:



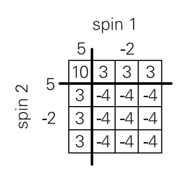
Geometric or area models will be useful to some students as these models provide a pictorial representation of the analysis which provides the students with a visual insight into the concept of probability. Consider the following situation. One of the events at your school's spring fair is a game of chance involving points. For each turn, a player spins and gets the points indicated in the area in that the spinner lands. Each player should add the numbers obtained by spinning twice. What are all the possible sums? What are the probabilities for obtaining each of these sums?



Students will notice that the spinner suggests that getting –2 will happen three-quarters of the time, while getting 5 will occur one quarter of the time. Using a grid of 16 squares to represent



the probability of 1, they would draw a vertical line (as in fig. 1) to represent the probabilities for the first spin (1/4 and 3/4). They would then separate the grid horizontally (as in fig. 2) to represent the probabilities of getting a –2 or 5 on the second spin. They would then analyse the grid to find the probabilities of obtaining the sums –4, 3, and 10.



P(–4) = 9/16, P(3) = (3/16)×2, and

P(10)= 1/16.

Now, using these results the students can be asked to create a situation where a player must accomplish something in order to win the game.

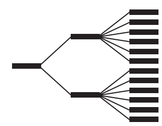
Probability

Worthwhile Tasks for Instruction and/or Assessment

G4

Pencil and Paper

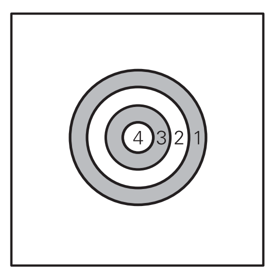
1) This incomplete tree diagram lists all the outcomes of tossing a coin and then rolling a die.



   a) Copy and complete the diagram.
   b) How many pairs of outcomes are there in this multiple event?
   c) What is the probability of tossing a head on the coin and then rolling a six on the die?
   d) What is the probability of tossing a head on the coin and then rolling an even number?
   e) What is the probability of *not* tossing a head on the coin and then rolling an even number of the die?
2) In a restaurant there are four kinds of soup, 12 entrees, six desserts, and three drinks. How many different four-course meals can a patron choose from? If 4 of the 12 entrees are chicken and two of the desserts involve cherries, what is the probability that someone will order wonton soup, a chicken dinner, a cherry dessert, and milk?
3) Licence plates for cars often have three letters of the alphabet, then three digits from 0 to 9. How many possible different licence plates can be produced? What is the probability of having the plate "CAR 000"?

Performance

4) The dart board at the right consists of four concentric circles whose centre is the centre of the square board. The side length of the square is 36 cm. The circles have radii 2 cm, 4 cm, 6 cm, and 8 cm respectively. A dart hitting the bull's eye or one of the shaded rings scores the indicated number of points. A hit anywhere else on the board scores 0 points. Assume that a dart thrown at random hits the board. Determine the probability of scoring:



   i) 4 points ii) 3 points iii) 2 points iv) 1 point v) 0 points
5) The following problem illustrates the usefulness of geometric probability. A tape recording is made of a meeting between a senator and her aide. Their conversation starts at the 21st minute on a 60-minute tape and lasts 8 minutes. While playing back the tape the aide accidentally erases 15 minutes of the tape.
   a) What is the probability that the entire conversation was erased?
   b) What is the probability that some part of the conversation was erased?
   c) Suppose the exact portion of the conversation on the tape is not known, except that it began sometime after the 21st minute. What is the probability that the entire conversation was erased?
6) Consider finding the area of the region bounded by the ellipse $4x^2 + y^2 = 4$. Enclose the ellipse in a rectangle whose sides pass through the x- and y-intercepts, and then consider the rectangular region to be a dart board. Suppose several darts thrown at random hit the rectangular region.
   a) Explain how probability can be used to approximate the area of the region bounded by the ellipse.
   b) Explain how probability can be used to approximate the area of the region bounded by the equation $y = -x^2 + 4$.

Suggested Resources

Probability

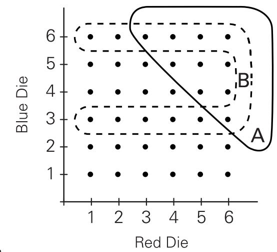
Outcomes

SCO: In this course, students will be expected to

G5 determine conditional
(111) **probabilities**

Elaboration—Instructional Strategies/Suggestions

G5(111) Ask the students a question such as "What is the probability that event A occurs if it is known that event B has occurred?" You should, through specific examples and some discussion, be able to get the class to arrive at a definition for conditional probability. For example:



If two dice, one red and one blue, are thrown and it is known that the blue die shows a number divisible by three, ask students what the probability is that the total on both dice is greater than 8? The condition that the number on the first die be divisible by three changes the sample space under consideration.

In particular, the new sample space contains only the 12 points shown inside the dashed closed curve at the right. For how many of these points is the total greater than 8? In light of the fact that all 36 points in the original sample space were assumed to be equally likely, students should agree that it seems reasonable to say that all 12 points in this sample space are equally likely. Then, given the condition that the number on the blue die is divisible by three, students should calculate the probability of having a total greater than 8 is equal to $\frac{5}{12}$.

For any two events A and B, the symbol "$P(A|B)$" is used to designate the probability that event A occurs given that event B has occurred. This is called a conditional probability because the condition is given that event B has occurred.

To evaluate $P(A|B)$ reconsider the above problem. Let the original sample space be the set of 36 possible outcomes shown in the diagram, let A be the set of points for which the total number of spots showing is greater than 8, and let B be the set of points for which the number of spots showing on the first die is divisible by three. Then $A \cap B$, pronounced 'A intersect B' consists of the 5 points indicated in the diagram by the triangular shape. In this case, to determine the conditional probability $P(A|B)$, divide the number of points in $A \cap B$ by the number of points in B. Of course, if the points of the original sample space were not equally likely, the result could not be obtained by simply counting points. Therefore, the probability of event A given that event B has occurred is defined as the probability of $A \cap B$ divided by the probability of B.

The probability that event A occurs if it is known that event B has already occurred is known as "conditional probability." It is symbolized as $P(A|B)$, and calculated using

$P(A|B) = \frac{P(A \cap B)}{P(B)} \leftarrow$ new sample space.

If the first of three tosses of a fair coin is heads, find the probability of getting exactly two heads in three tosses.

Solution: Let A be the event "getting exactly two heads."

Let E be the event "getting a head on the first throw."

$$(A \cap E) = \{HHT, HTH\}$$

so, $P(A \cap E) = \frac{2}{8} = \frac{1}{4}, P(E) = \frac{1}{2}$

$$\therefore P(A|E) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

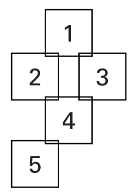
Probability

Worthwhile Tasks for Instruction and/or Assessment

G5(111)

Performance

1) What is the probability of getting two fives when two dice are thrown and it is known that at least one landed with a five up?
2) Assuming the probability of being born male is 0.5. In a family of three children it is known that at least one child is male. What is the probability that all three children are male?
3) A weather report indicates an 80% probability of rain on Monday, 60% on Tuesday, and 20% on Wednesday. What is the probability that it will rain on at least one of the three days?
4) In the MAKE-A-NUMBER game, you draw a **Condition Card**. Then you draw two **Number Cards** from a stack of only five cards and place them side-by-side to make a two-digit number. If the two-digit number fits your **Condition Card**, you score one point.



1. Condition The number is divisible by 3.	2. Condition The sum of the digits of the number is 5.
Probability:________	Probability:________
3. Condition The number is greater thatn 40.	4. Condition The number is a prime number.
Probability:________	Probability:________
5. Condition The tens digit of the number is greater than the ones.	6. Condition The units digit of the number is divisible by the tens digit
Probability:________	Probability:________

Determine the probability of scoring with these **Condition Cards**.

Suggested Resources

Shulte, Albert P., ed. *Teaching Statistics and Probability. 1981 Yearbook*. Reston, VA: NCTM, 1981.

Probability

Outcomes

SCO: In this course, students will be expected to

G5 (111) determine conditional probabilities

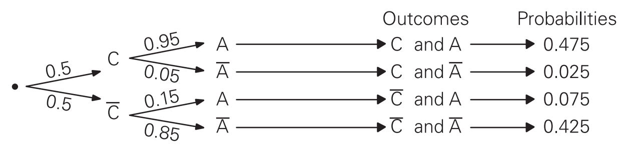
Elaboration—Instructional Strategies/Suggestions

G5(111) Tree diagrams are often used to organize all the possible combined outcomes of a multiple event. For example: A fast food outlet requires prospective employees to take an employability exam as a basis for hiring. They estimate that 50% of the applicants would complete a full year's work. They also estimate that 15% of the unqualified applicants pass the exam and 5% who are qualified fail the exam. All who pass the exam are hired and none of the other are. What fraction of applicants are hired? If an applicant passes the exam, what is the probability that he or she will complete a year's work?

Let C represent the event "applicant completes a year"

Let A represent the event "applicant passes exam"

Thus $\overline{C}$ and $\overline{A}$ are the complementary events.



The first and third paths represent hired students, so $P(A) = .475 + .075 = .55$. So 55% of the applicants are hired. The first path shows 47.5% of the applicants are both hired and work a year, and so the conditional probability (see previous two-page spread) of completing a year, given the passing the exam and being hired, is $P(C|A) = P(C \cap A) / P(A) = .475 / .55 = 0.86$.

So, 86% of those hired complete a year's work. An area diagram example: Suppose that Tom is a 60% free throw shooter in basketball. At the end of a game he was fouled and his team is losing by two points. He will shoot "one-and-one." What is the probability that he misses the second shot? To solve this problem, students could use an area model like that on the right. The probability of making the first shot is shown in fig. 1, then if he makes the first shot, he gets the second shot. Fig. 2 shows the probability of missing the second shot.

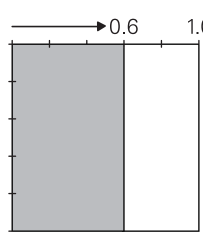


fig. 1

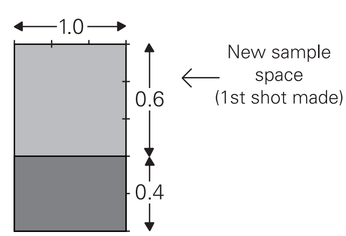


fig. 2

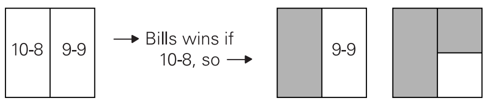
Probability

Worthwhile Tasks for Instruction and/or Assessment

G5(111)

Performance

1) Two gamblers play a game for a stake that goes to the first player to gain 10 points. If the game is stopped when the score is 9 to 8, in favour of Bill, what is the probability that Bill will win when the game is resumed? Use an area model to help ... (It is assumed that both players have equal chances of winning each point.)
   If the score is 9–8 then the next score will be ...



   If the game goes to 9–9, either one might win.
   a) What can you conclude from this?
   b) What would be the solution to the problem if Bill was winning 9–7 when the game is stopped?

2) As archers, Rita hits the target $\frac{2}{5}$ of the time and David $\frac{1}{3}$, of the time. They are going to have a contest with David shooting first. They alternate shots until one wins by hitting the target. Who is favoured? What is each contestant's probability of winning?

3) A certain restaurant offers select-your-own desserts. That is, a person may select one item from each of the categories listed:

Ice Cream	**Sauce**	**Extras**
vanilla	chocolate	cherries
strawberry	caramel	peanuts
chocolate mint		

   a) Using a tree diagram, list all possible desserts that can be ordered.
   b) Would you expect the choices of a dessert to be equally likely for most customers?
   c) If the probability of selecting chocolate ice cream is 40%, and vanilla is 10%, chocolate sauce is 70%, and cherries 20%, describe the dessert with the highest probability of being selected.

4) A certain model of automobile can be ordered with one of three engine sizes, with or without air conditioning, and with automatic or manual transmission.
   a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.
   b) Suppose you want the car with the smallest engine, air conditioning, and manual transmission. A General American agency tells you there is only one of the cars on hand. What is the probability that it has the features you want, if you assume the outcomes to be equally likely?

5) Jennifer dresses in a skirt and a blouse by choosing one item from each category.

Skirts
tan plaid gray
stripe 1 stripe 2
stripe 3

Blouses
white pink 1 pink 2
red

   a) Show, by means of a tree diagram, all the outfits she can make if one has three striped skirts and two pink blouses and only one of everything else.
   b) What's the probability of her wearing something striped and white knowing that she already has a striped skirt on?

Suggested Resources

Newan, Claire et al. *Exploring Probability. Quantitative Literacy Series.* White Plains, NY: Dale Seymour Publications, 1987.

Probability

Outcomes

SCO: In this course, students will be expected to

$G1_3$ develop and apply simulations to solve problems

Elaboration—Instructional Strategies/Suggestions

$G1_3$ Simulation is a procedure developed for answering questions about real problems by running experiments that closely resemble the real situation.

Suppose the students want to find the probability that a three-child family contains exactly one girl. If students cannot compute the theoretical answer and do not have the time to locate three-child families for observation, the best plan might be to simulate the outcomes for three-child families. One way to accomplish this is to toss coins to represent the three births. A head could represent the birth of a girl. Then, observing exactly one head in a toss of three coins would be similar, in terms of probability, to observing exactly one girl in a three-child family. Students could easily toss the three coins many times to estimate the probability of seeing exactly one head. The result gives them an estimate of the probability of seeing exactly one girl in a three-child family. This is a simple problem to simulate, but the idea is very useful in complex problems for which theoretical probabilities may be nearly impossible to obtain.

Students need experience thinking through complete simulation processes. When choosing a simple device to model the key components in the problem they have to be careful to choose a model that generates outcomes with probabilities to match those of the real situation. Students could use devices such as coins, dice, spinners, objects in a bag, and random numbers.

Students need to understand that the experimental probability approaches the theoretical probability as the number of trials increases. They should also realize that knowing the probability of an event gives them no predicting power as to what the outcome of the next trial will be. However, after enough trials, they should be able to predict with some confidence what the overall results will be.

When conducting simulations students should follow a certain process such as the one outlined: (see next page for an actual class activity).

Step 1: State the problem clearly.

Step 2: Define the key components.

Step 3: State the underlying assumptions.

Step 4: Select a model to generate the outcomes for a key component.

Step 5: Define and conduct a trial.

Step 6: Record the observation of interest.

Step 7: Repeat steps 5 and 6 until 50 trials are reached.

Step 8: Summarize the information and draw conclusions.

Probability

Worthwhile Tasks for Instruction and/or Assessment

G1$_3$

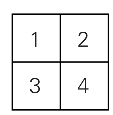
Pencil and Paper

1) Consider the following problem:
   Marie has not studied for her history exam. She knows none of the answers on the seven-question true-and-false section of the test. She decides to guess at all seven. Estimate the probability that Marie will guess the correct answers to four or more of the seven questions.
   Ask students to complete the following:
   a) What are you being asked to do?
   b) To perform a simulation, what assumptions should you make?
   c) Describe the model you would choose to perform the simulation.
   d) Pretend that you are watching the simulation. Describe what you observe for the entire simulation.
   e) What conclusion do you think would be made?

2) Suppose a stick or a piece of raw spaghetti has been broken at two random points. What is the probability that the three pieces will form a triangle? (The pieces must touch end to end.)
   a) Describe the process that might be used to estimate the answer using experimental probability.
   b) Instead, Robert is going to use a simulation. He assumes the spaghetti is 100 units long, and he is going to generate two random numbers between 0 and 100 using each as a side of a triangle. How would Robert find the third side? How would Robert check to see if the numbers represent the lengths of the side of a triangle?
   c) Perform this simulation to find the answer.

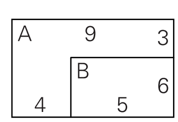
Performance

3) Dale, a parachutist, jumps from an airplane and lands in a field. What are the chances that Dale will land in a particular numbered plot? Make a field grid using a normal sheet of graph paper divided into four equal areas.



   a) Model the situation by tossing a thumbtack onto the grid from a metre or more away. (If the tack bounces off the sheet—don't count it as a toss.)
      In your response consider several questions:
      Is there an equal chance to land in each plot?
      How many times did Dale land in plot 1?
      Discuss the experimental probability results versus the theoretical probability results for the given field.
   b) Conduct the experiment again, but use a field divided into plots A and B to find the probability that Dale will land in Plot A.
   c) Perform a simulation to answer the same problem as in (b). Compare the results of the simulation with that of the experiment. Comment.

4) Perform simulations to solve the following problems:
   a) What is the probability that all five children in a family will be girls?
   b) A couple leaves for work anytime between 7:00 and 8:00 am. Their newspaper arrives any time between 6:30 and 7:30 am. What is the probability that they get the paper before they leave for work?



Suggested Resources

Zawojewski, Judith. *Dealing With Data and Chance. Curriculum and Evaluation Standards for School Mathematics Addenda Series. Grades 5–8.* Reston, VA: NCTM, 1992.

Probability

Outcomes

SCO: In this course, students will be expected to

G7 distinguish between situations that involve permutations and combinations

Elaboration—Instructional Strategies/Suggestions

G7 Before describing different situations in terms of permutations and combinations, students need to have an opportunity to solve simple counting problems (see elaboration for $G2_3$, p. 84). They may wish to organize their work into systematic lists and/or tree diagrams. As the number of choices increases, they will see the need for a way to count more efficiently. For example:

a) How many different routes can you take from Sydney to Halifax through Antigonish?
b) How many routes are there from Antigonish to either Halifax or Sydney?

Following this, the class might be split into two groups—one will do Problem A, the other Problem B. Students should present their solution to the class.

Problem A: Suppose there were three people, Adam, Marie, and Brian, standing in line at a banking machine. In how many different ways could they order themselves?

Problem B: The executive of the student council has five members. In how many ways can a committee of three people be formed?

Solutions might look like:

Problem A: using a systematic list:A M B, A B M, M B A, M A B, B A M, B M A

Problem B: using a systematic list : if Adam, Marie, and Brian along with Dennis and Elaine were on the executive, then to select committees of three, starting with Adam, Marie and Brian, the five permutations in the answer to A above would result in the same five people being the committee, so they represent one combination.

The essential difference between these two situations needs to be discussed and emphasized. Eventually, Problem A should be described as a permutation (order is important), Problem B as a combination (order not important).

Probability

Worthwhile Tasks for Instruction and/or Assessment

G7

Pencil and Paper

1) For each of the following, decide whether permutations or combinations are involved.
   a) The number of committees of two that can be formed from a group of 12 people.
   b) The number of possible lineups for a baseball team that can be formed from 12 people without regard to position (a baseball team consists of nine players, as follows: pitcher; catcher; first, second, and third basemen; shortstop; right, centre, and left fielders).
   c) The number of five-letter licence plates that can be formed from 12 different letters.
   d) The number of six subsets that can be formed from 12 different letters.
   e) The number of five-man basketball teams that can be formed from 10 players.
   f) The number of ordered triples that can be formed from 10 different numbers.
   g) The number of ordered triples that can be formed from the numbers 1, 1, 1, 3, 3, 5, 5, 5, 5, and 4.

2) The manager of a baseball team needs to decide the batting order for the season opener. In how many ways can the first four batters be arranged on the batting roster? Is this a permutation or combination question? Explain.

3) As a promotion, a record store placed 12 tapes in one basket and 10 compact discs in another. Pierre was the one millionth customer and was allowed to select 4 tapes and 4 compact discs. To find how many selections that can Pierre make, does one use permutations or combinations? Explain.

4) Three identical red balls (R) and two identical white balls (W) are placed in a box. How many ways are there of selecting the balls in the following order?

   RWRRW

5) a) Find the total number of arrangements of the letters of the word "SILK."
   b) Find the total number of arrangements of the letters of the word "SILL."
   c) How are your answers in a) and b) alike? How are they different?

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

A6 **develop an understanding of factorial notation and apply it to calculating permutations and combinations**

G8 **develop and apply formulas to evaluate permutations and combinations**

Elaboration—Instructional Strategies/Suggestions

A6 As students refine their methods of counting, moving from tree and area diagrams and listing through the fundamental counting principles, they should learn to recognize and use $n!$ (n factorial) to represent the number of ways to arrange n distinct objects. For example, the product rule can be used to find the number of possible arrangements for three people standing in a line. There are three people to choose from for the front of the line. For each of these choices, there are two people to choose from for the second position in the line. For each of these choices, there is one person to choose from the end of the line. Therefore, there are $3 \times 2 \times 1$ or six possible arrangements.

In another example, at a music festival, eight trumpet players competed in the Baroque class. After the judging, they were awarded first, second, third... down to eigth place. In how many ways could their placements be awarded?

If all the trumpet players were given a position first, second, third, ... , eighth, then the total number of possible standings could be calculated by using reasoning like: There are eight people eligible for first, which leaves seven eligible for second, six people for third ... leading to a calculation $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. This product can be written in a compact form as 8! and is read "eight factorial."

In general, $n! = n(n-1)(n-2)...(3)(2)(1)$, where $n \in N$ and $0! = 1$.

A6/G8 If there are only three prizes to be given to the 8 trumpeters, how many ways could placement be awarded?

Students should reason that eight people could come first, only seven could come second, and six could come third $\rightarrow 8 \times 7 \times 6 \rightarrow 336$. This could be worded "How many permutations are there of eight distinct objects taken three at a time?"

The symbol commonly used to represent this is ${}_8P_3$, or ${}_nP_r$ for the number of "n" objects taken "r" at a time. Students should notice that

$${}_8P_3 = 8 \times 7 \times 6$$

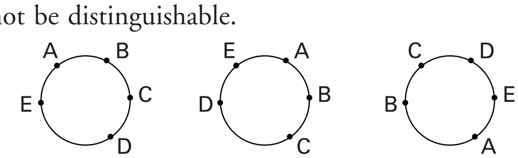
$$also,\ {}_8P_3 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

$$so,\ {}_8P_3 = \frac{8!}{5!}$$

$${}_8P_3 = \frac{8!}{(8-3)!}$$

$$so,\ {}_nP_r = \frac{n!}{(n-r)!}$$

Students should note that when five people are to be arranged in a straight line there would be 5! or 120 ways to do this. However, if the same five people were to be arranged around a table in the order, say A, B, C, D, and E, their relative position to each other would not be distinguishable.



Thus, the total number of arrangements would be:

$$\frac{{}_5P_5}{5} = \frac{5!}{5} = 4! = 24$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

A6

Pencil and Paper

1) The town of Karsville, which has 32 505 automobiles, is designing its own licence plates for residents to place on the front of their automobiles.
   a) Ask students to use counting principles to determine the best of the following three options and explain their choice:
      i) a licence made from using four single-digit numerals from 1 to 9
      ii) a licence made of three single-digit numerals from 1 to 9 and one letter from the alphabet
      iii) a licence made from three single-digit numerals from 1 to 9 and two letters from the alphabet.
   b) Ask students to select the best combination of single-digits from 1 to 9 and letters from the alphabet to suit the purposes of this town and to defend their selection.

2) The figure shows three black marbles and two white marbles.
Suppose they are in a box. Without looking in the box, choose two of the five marbles. How many ways are there to select two marbles that are the same colour? Each a different colour?

A6/G8

Pencil and Paper

a) Indicate which of the following are true (T) and which are false (F).
   i) $\frac{5!}{4!} = 5 \times 4$
   ii) $10 \times 9 \times 8 = \frac{10!}{7!}$
   iii) ${}_8P_2 = 56$
   iv) ${}_{100}P_4 = 100 \times 100 \times 99 \times 98 \times 97$

b) Create a story where each expression above would be used in the solution.

7) There are five points, no three of which are collinear, on a plane.
   a) How many segments can be formed using these five points as endpoints?
   b) If consecutive points are joined, a convex polygon is formed. How many diagonals does this polygon have?

8) A local pizza restaurant has a special on its four-ingredient 20 cm pizza. If there are 15 ingredients from which to choose, how many different "specials" are possible?

9) Explain why the following theorem would be true:

   A circular arrangement of 'n' items can be calculated using: $\frac{{}_nP_n}{n} = (n-1)!$

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

A6 **develop an understanding of factorial notation and apply it to calculating permutations and combinations**

G8 **develop and apply formulas to evaluate permutations and combinations**

G7 **distinguish between situations that involve permutations and combinations**

Elaboration—Instructional Strategies/Suggestions

A6/G8 Refer back to the problem where there are five members on the executive of the student council. If these five were elected from a list of 10 candidates for executive position, such as president, vice president, secretary, the number of ways 10 people can be slotted into five positions would be found using permutations

${}_{10}P_5 = \frac{10!}{(10-5)!} = 30240$.

A6/G8/G7 From these five people a committee of three is struck. If the five people are represented by *A*, B, C, D, and E, then clearly a committee with A, B, and C is the same as a committee with C, A, and B. So, the order of the selection is not important and the arrangement is called a combination. Therefore, since ABC, ACB, BAC, BCA, CAB, and CBA are all considered the same committee, they represent only one committee of three selected from the five people. The number of permutations of A, B, and C is 3!. Thus, the number of committees from the original list of 10 candidates

$$= \frac{\text{number of ways the executive was chosen}}{3!}$$

$$= \frac{30240}{3!}$$

$$= 5040$$

That is ${}_{10}C_3 = \frac{{}_{10}P_3}{3!} = 5040$

and the number of committees from the five member executive selected would be

${}_5C_3 = \frac{{}_5P_3}{3!} = 10$.

A combination of "*n*" objects taken "*r*" at a time is any subset of size "*r*" taken from the "*n*" objects. The number is denoted by $\binom{n}{r}$ (read "*n*" choose "*r*"), or ${}_nC_r$.

The number $\binom{n}{r}$ can be evaluated by investigating the connection between permutations and combinations.

Since ${}_nP_r = \frac{n!}{(n-r)}$. Thus, in general, $\binom{n}{r} = \frac{{}_nP_r}{r!}$. $\therefore {}_nC_r = \frac{n!}{r!(n-r)!}$

For example: A committee of size 4 and a committee of size 3 are to be assigned from a group of 10 people. How many ways can this be done if no person is assigned to both committees? Solution: First committee ${}_{10}C_4 = \binom{10}{4} = 210$ ways, and there are 6 people left for the second committee. Second committee ${}_6C_3 = \binom{6}{3} = 20$ ways.

Therefore the two committees can be assigned $210 \bullet 20 = 4200$ ways. Note: If the smaller committee was selected first then $\binom{10}{3}\binom{7}{4} = 120 \bullet 35 = 4200$ ways.

Probability

Worthwhile Tasks for Instruction and/or Assessment

A6/G8/G7

Pencil and Paper

1) a) Which of the following will produce the number of greatest magnitude? (Use estimation first.) Which will produce the smallest ?

i) $6!$ iv) $3! \div 4$ vii) $\frac{9!}{7!}$

ii) $11!$ v) $\frac{9!}{2!}$ viii) $\frac{100!}{2!}$

iii) $\frac{15!}{12!}$ vi) $\frac{9!}{2!}$ ix) $4! - 3!$ x) $\frac{7!}{6!}$

b) Pick three of the above expressions and create a problem in which these symbols would be used in the solution.

A6

Pencil and Paper

2) Write each as a ratio of factorials.

a) $7 \times 6 \times 5$ d) $30 \times 29 \times 29 \times 12 \times 11 \times 10 \times 9$

b) $19 \times 9 \times 8 \times 7 \times 5 \times 19$ e) $20 \times 19 \times 18 \times 17$

c) $10 \times 9 \times 8 \times 7 \times 6$ f) $\frac{50 \times 49 \times 48 \times 47 \times 46}{5!}$

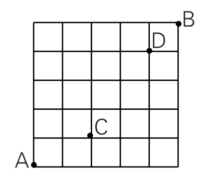
G8/G7

Performance

3) A government committee of size 9 is to be selected from five liberals, four reformers, and four new democrats. How many ways can this be done it each of the three parties must be equally represented?

4) Explain in words why you think a combination lock is called a combination lock instead of a permutation lock.

5) A fly goes from A to B in the grid by travelling only to the right or upwards. How many possible routes are there? How many routes are there that go through C, but not through D?



6) Linda, Gino, and Sam each draw 3 cards from a deck of 52 playing cards and do not replace them.
   a) If Linda goes first, in how many ways can she pick 3 cards?
   b) In how many ways can Gino draw his cards after Linda has drawn hers?
   c) Finally, in how many ways can Sam draw her cards?

7) A quarterback on a football team has seven different plays to use in a game. In order to confuse the defence of the other team, the quarterback does not want to repeat the same sequence of plays too often. How many different sequences of three plays has she to choose from if no play is repeated?

8) Mr. Burble teaches 182 students mathematics at Harry High. He tells his students that they must do these six problems, but that they can do them in any order. Is it possible for each of his students to do them in a different order? Explain.

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

B8 **determine probabilities using permutations and combinations**

G8 **develop and apply formulas to evaluate permutations and combinations**

Elaboration—Instructional Strategies/Suggestions

B8/G8 Students should now apply ${}_nP_r$ and ${}_nC_r$ to probability problems.

One practical use of permutations and combinations is in the field of probability. For example, a deck of 52 cards is shuffled well. What is the probability that A, K, Q of spades will be dealt to you as the first three cards?

The students would reason that since they want to see a particular three cards from 52 possible cards, they would use ${}_nP_r$ or ${}_{52}P_3$.

$${}_{52}P_3 = \frac{52!}{(51-3)!} \Rightarrow \frac{52!}{49!} \Rightarrow 132600$$

and only one of those outcomes is favourable, so

$$P(A,K,Q) = \frac{1}{132600}$$

Combinations are sometimes used along with other counting techniques. For example, suppose that a 17-member student council at the high school consists of 9 girls and 8 boys. A committee of 4 council members must have 2 girls and 2 boys. There are ${}_9C_2$ or 9!/(7!2!)=36 ways of selecting the 2 girls, and ${}_8C_2$ or 8!/(6!2!)=28 ways of selecting 2 boys. Because the committee must include 2 girls and 2 boys, there are $36 \times 28 = 1008$ ways of forming the committee. If the 4 committee members are selected at random, the probability that the committee will consist of 2 boys and 2 girls is 1008/2380, or about 0.424.

Probability

Worthwhile Tasks for Instruction and/or Assessment

B8/G8

Performance

1. Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random. Assume all players are equally qualified for every position.

   (a) In how many ways could the positions be filled?

   (b) What is the probability that Duffy will be the pitcher?

   (c) What is the probability that David, George or Duffy will be first baseman?

   (d) What is the probability that David, George or Duffy will be first baseman and Eleanor or Georgina will be pitcher?

2. The numbers on a raffle ticket contain three digits. The first digit cannot be zero.

   (a) What is the probability of ticket number 917 winning the grand prize? What assumption did you make?

   (b) What is the probability that a ticket with three as a second digit wins the grand prize?

3. If a coin is flipped five times, what is the probability of flipping a sequence that contains four heads and one tail?

4. (a) Give a detailed mathematical reason why Nova Scotia changed their automobile licence plates to ones that contain three letters of the alphabet and three while numbers less than ten.

   (b) What would be the probability of having licence plate that reads MTH 101.

5. Fran ran out and bought 20 6/49 tickets when she heard that the jackpot was $10 000 000. Anna told her she was nuts. Fran says, "If I only buy one ticket I only have one chance of winning, but if I buy 20 tickets I have 20 chances." How should Anna respond to help Fran understand both her chances of winning, and how wise is her investment?

Suggested Resources

Hannah, Frank L., "Probability," Venture Publishing, 1993.

Forrster, Paul A. "Algebra and Trigonometry," Addison-Wesley, 1994.

Probability

Outcomes

SCO: In this course, students will be expected to

G10 connect Pascal's Triangle with combinatorial coefficients

G9 demonstrate an understanding of binomial expansion and its connection to combinations

Elaboration—Instructional Strategies/Suggestions

G10 Students should be asked to take binomials like $(x + y)$ and find simplified expressions for $(x + y)^2$, $(x + y)^3$, $(x + y)^4$, etc., and look for patterns in their coefficients. They should be able to find a connection between the expansion power and that same row in the Pascal's Triangle with respect to the coefficient values. $\left((x+y)^0\right)=1$ is the top row (row 0).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . .

Pascal's Triangle

G9 The counting techniques, discussed in G3, p. 142, can be useful in the multiplying of polynomials. Looking at the product of $(a + b)(c + d) = ac + bc + ad + bd$, students should notice that each term in the expansion has one factor from $(a + b)$ and one factor from $(c + d)$. e.g., ac has two factors a and c. The a is from $(a + b)$ and the c is from $(c + d)$. Thus the number of terms in the expansion is four since there are two choices from $(a + b)$ and two choices from $(c + d)$. Students should also notice that since there are two factors $(a + b)$, and $(c + d)$ there are two factors in each term of the expansion.

$$(a+b)(c+d) \rightarrow ac+bc+ad+bd \rightarrow 4 \text{ (each term has two factors)}$$

$\uparrow$

two factors
in each term

The product of one binomial and itself follows the same pattern. $(x+y)^2 \rightarrow (x+y)(x+y) \rightarrow xx+xy+yx+yy$, but the multiplication would be completed by collecting the like terms and using exponents: $x2 + 2xy + y2$. Students should consider $(x+y)^5 = (x+y)(x+y)\ldots(x+y) \rightarrow xxxxx + xxxyy + \ldots + yyyyy$. Each term is made up of five factors and using exponents will look like $x^a y^b$ where

$$a+b=5\text{, e.g., }\quad \begin{aligned} &xxxxx \rightarrow x^5 \rightarrow x^5y^0 \rightarrow 5+0=5 \\ &xxxy \rightarrow x^4y \rightarrow x^4y^1 \rightarrow 4+1=5 \\ &xxxyy \rightarrow x^3y^2 \rightarrow 3+2=5 \end{aligned}$$

G10/G9 In collecting the like terms, how many terms will be made up of the two factors x^2y^3? To answer this students should count the number of ways to make x^2y^3, e.g., the two factors of x must come from two of the five factors in each term of $(x + y)^5$. This can be done $\binom{5}{2}$ or ${}_5C_2$ = 10 ways. The three factors of y must come from the remaining three factors in each term of $(x + y)^5$ and this can be done in only one way. So the coefficient of x^2y^3 (${}_5C_3$) will be 10. Students should note that these coefficients are values in the fifth row of Pascals' Triangle.

Students should examine the pattern changes in the signs between terms when $(x – y)^5$ is expanded. Because the second term in the expression $(x – y)^5$ could be considered negative $(–y)$, then the terms in the expansion that have odd numbers of y-factors will be negative. When exponents or coefficients are included in the binomial to be expanded $(x^2 + 3y)^3$ students should be aware that for every x-factor, there is now an x^2-factor, and for every y-factor there is now a $3y$-factor, e.g., when x is replaced with x^2 and y with $3y$ the expansion becomes:

$$(x^2-3y)^3 = \binom{3}{3}(x^2)^3 - \binom{3}{2}(x^2)^2 3y + \binom{3}{1}(x^2)(3y)^2 - \binom{3}{0}(3y)^3$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil **(G10/G9)**

1) What is the coefficient of *the* x^4y^2 term in each of the following?
   a) $(x + y)^6$
   b) $(x - 2y)^6$
   c) $(2x + y)^6$
   d) $(3x - 2y)^6$

2) When examining the terms from left to right, find the specified term in each expansion.
   a) 10 th in $(x - y)^{12}$
   b) 20 th in $(2x - 1)^{19}$
   c) 8 th in $(a + b)^{10}$
   d) 2 nd in $(x^3 - 5)^7$
   e) 3 rd in $(1 - 2x)^9$
   f) 15 th in $(1 + a^2)^{24}$

3) a) Find the sum of the elements in each row, for the first six rows of Pascal's Triangle.
   b) Find the number of subsets in a 0–, 1–, 2–, 3–, 4–, and 5– element set.
   c) How are parts (a) and (b) related?
   d) How many elements are there in an n-element set?

4) Find a decimal approximation for 1.02^{10} by writing it as $(1 + 0.02)^{10}$ and calculating the first five terms of the resulting binomial series.

5) Henrietta is expanding $(3a - 2b^2)^3$. In her work below, explain what she is doing when going from step 2 to step 3. Is her work correct? Explain. What should she do to complete her work?

   step 1: $a \quad a\ b \quad a\ b \quad b$

   step 2: $a^3 \quad a^2\ b \quad a^1\ b^2 \quad b^3$

   step 3: $(3a)^3 \quad (3a)^2\ (2b^2) \quad (3a)^3\ (2b^2)^2 \quad (2b^2)^3$

G10/G9

Journal

6) Betty Lou missed math class today. Helen phoned her at night to tell her about how combinations are helpful when expanding binomials. Write a paragraph or two about what Helen would have told her.

7) When expanding $(a^2 - 2b)^5$, Wally gets confused about the exponents in his answer. Write a paragraph to Wally to help him remember how to record the exponents on this expansion.

Suggested Resources

Probability

Outcomes

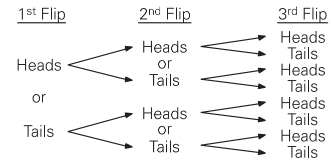
SCO: In this course, students will be expected to

G11 (111) connect binomial expansions, combinations, and the probability of binomial trials

$G1_3$ develop and apply simulations to solve problems

Elaboration—Instructional Strategies/Suggestions

G11(111) Many experiments consist of more than two parts, and if these parts are independent of one another, students can use the concept of a product model or tree diagram to help them with their counting and probability calculations. For example, when flipping a fair coin three times, a tree diagram determines that the sample space has eight outcomes. The probability of any one of them being selected is 1/8.



P (of any one of eight outcomes) = $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

$G1_3$/G11(111) The tossing of three coins discussed above could be used as a simulation model. Say the students want to solve the problem "in a family with three children what is the probability that the first two children are girls and the third is a boy."

Experiments that consist of repeated trials of a simple experiment (e.g., tossing a coin) using a model with only two possible outcomes (heads or tails) are called binomial trials. Suppose students needed to find the probability of getting exactly three heads in 10 tosses of a fair coin. Since each trial has two choices there would have to be 2^{10} = 1024 branches on a tree diagram. The answer would be the sum of all the probabilities of the branches that contain three heads and seven tails. The number of ways 3 heads and 7 tails could be arranged is the "ten choose three ($_{10}C_3$)" or $\binom{10}{3}$, and so the probability of this happening would be $\frac{\text{\# of success}}{\text{total number of outcomes}} = \frac{_{10}C_3}{2^{10}}$.

Another way to consider this is that the probability of any one of the 2^{10} branches being selected would be the product of $\frac{1}{2}$ for every H and $\frac{1}{2}$ for every T or $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$. Since every branch with three heads and seven tails has the same probability the answer is the number of these branches times the probability for each branch. The number of branches will be the number of ways of choosing the three heads out of ten tosses $\binom{10}{3}$. Hence, $\binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$. Students should compare $\frac{\binom{10}{3}}{2^{10}}$ and $\binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$ to see how one is the same as the other.

In general, in binomial trials there are two outcomes for each of n trials. One of the two outcomes is a "success," the other a "failure." These are labelled p and q respectfully and $q = 1 - p$. The number of successes in n trials is labelled s. Thus the probability of getting s successes and $n - s$ failures in n binomial trials is $\binom{n}{s}p^s q^{n-s}$. For example, suppose students conduct an experiment of flipping a coin. The coin is bent, so the probability of heads (success) is 0.3. If they flip the coin five times, what is the probability of three tails and two heads? Students should now be able to answer this with $P(3T,2H) = \binom{5}{2}(0.3)^2(0.7)^3$. Students should recognize this as a term in the binomial series that comes from expanding $(0.7 + 0.3)^5$. Students might want to use the 'randBin' feature on their calculators, or other software technology to conduct experiments or simulations where random samples are needed from populations that include only two possible outcomes (e.g., yeses, and nos).

Probability

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Worthwhile Tasks for Instruction and/or Assessment

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* G11(111)/G1$_3$

Performance

1) Find the probability of getting 10 heads in 15 throws of a bent coin if the probability of heads on the bent coin is $\frac{2}{3}$.
2) Find the probability of getting exactly two ones in six rolls of a fair die.
3) If $n = 4$ and $p = \frac{1}{2}$, for what value of s will $\binom{n}{s} p^s q^{n-s}$ be largest? Answer the same question for $n = 4$ and $p = \frac{1}{3}$ and for $n = 5$ and $p = \frac{2}{3}$.
4) If Jamie is serving he wins a tennis game against Sam with probability $\frac{4}{5}$, but if he is receiving he wins with probability $\frac{2}{5}$. Jamie and Sam agree to play five games, and Jamie bets that he can win two in a row. If Jamie wins the toss, should he elect to serve or receive? Draw two tree diagrams and verify your answer.
5) A teacher made up a fair 10-item true and false test. Kira missed a few days just before the test and thought if she answered the questions randomly selecting *T*s and *F*s, she might do allright. When she was done she had 4 *T*s and 6 *F*s. What is the probability that Kira's 4 *T*s and 6 *F*s are correct. Show how to find the answer two ways.

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Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

G12 demonstrate an
(111) **understanding of and solve problems using random variables and binomial distributions**

Elaboration—Instructional Strategies/Suggestions

G12(111) Once students have developed the pattern described on the previous two-page spread $\binom{n}{s}p^s q^{n-s}$, they can use it to calculate the probabilities of other related events: For example:

Let x = number of times the bent coin is "heads" in five flips.
Let $P(x)$ = probability that it is "heads" x times.
Therefore, with the probability of heads being 0.30 ...

no head: $P(0)=\left(\binom{5}{0}(0.3^0)(0.7^5)=0.16807\right)$

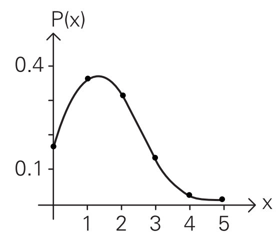
one head: $P(1)=\binom{5}{1}(0.3^1)(0.7^4)=0.36015$

: : :

five heads: $P(5)=\binom{5}{5}(0.3^5)(0.7^0)=0.00243$

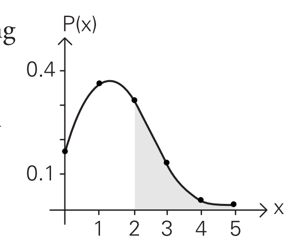
As a check on the answers, students should realize that x is certain to take on one of the values 0 through 5. So P (0 or 1 or 2 or 3 or 4 or 5) must equal 1 or 100%.

The independent variable x is called a random variable since you cannot be sure what value x will have on any one run of the random experiments. The dependent variable *P(x)* is the probability that the value is x. So P is a function of a random variable. The graph of *P(x)* for the above situation is shown.



The function P shows how the total probability, 1.00000, is "distributed" among the possible values of x. This function of a random variable is often called a probability distribution.
Since this particular distribution has probabilities that are terms of a binomial series, it is called binomial distribution. It is skewed left since the probability of heads is only 0.3.

Binomial distributions occur when students perform a random experiment repeatedly, and each time there are only two possible outcomes (e.g., heads or tails, boy or girl, win or lose, yes or no). Students have already learned that a normal distribution is the result of recorded measurement of the same phenomena repeated over and over and over again. Since the binomial distribution is the result of a very similar action or fact repeated over and over and over again, it would be expected that it too would approach a normal distribution if the given probability is 0.5. This can be simulated quickly using the 'randBin' feature of the graphing calculator or other software technology.



For example: randBin $(10,0.5,10)\rightarrow L_1$. Once they have found the probability distribution, they can use the properties of probability to calculate the probabilities of related events. For example, if the bent coin is flipped five times, as above, then the probability of getting at least two heads is

$P(x\geq 2)=P(2)+P(3)+P(4)+P(5)$.
$=0.3087+0.1323+0.02835+0.00243$

Probability

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Worthwhile Tasks for Instruction and/or Assessment

*** ** * G12(111)

Performance

1) **Heredity Problem:** If a dark-haired mother and a dark-haired father have a recessive gene for light hair, there is a probability of then having a light-haired baby. For this to happen, each must have a large x (dark hair) and a small x (light hair) gene. In order for the baby to be light-haired, it must have two small x genes
   a) What is their probability of having a dark-haired baby?
   b) If they have three babies, calculate P (0), P (1), P (2), and P (3), the probabilities of having exactly 0, 1, 2, and 3 dark-haired babies, respectively.
   c) Show that your answers to part b are reasonable by finding their sum.
   d) Plot the graph of the probability distribution, P.

2) **Multiple Choice Test Problem:** A short multiple choice test has four questions. Each question has five choices, exactly one of which is right. Willie Makitt has not studied for the test, so he guesses at random.
   a) What is his probability of guessing any one answer right? Wrong?
   b) Calculate his probabilities of guessing 0, 1, 2, 3, and 4 answers right.
   c) Perform a calculation that shows your answer to part b is reasonable.
   d) Plot the graph of the probability distribution in part b.
   e) Willie passes the test if he gets at least three answers right. What is his probability of passing?

4) What is the probability of getting exactly 50 heads when 100 coins are tossed?

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Suggested Resources

Probability

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Outcomes

SCO: In this course, students will be expected to

G11 (111) connect binomial expansions, combinations, and the probability of binomial trials

G12 (111) demonstrate an understanding of and solve problems using random variables and binomial distributions

Elaboration—Instructional Strategies/Suggestions

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* **G11(111)/G12(111)** Using the ideas developed over the last two two-page spreads, students can investigate some of the claims typically made in television and newspaper advertising. For example, a television commercial states that 8 out of 10 cats prefer Purrfect Chow. The claim is based upon a particular test in which 8 out of 10 cats chose Purrfect when given a choice between it and another cat food. A complaint is made by a rival cat food manufacturer. They say that 8 out of 10 would not be unusual, if it is assumed that cats have no particular preference for *Purrfect*. Assuming that cats will choose equally between one food or another randomly, what is the probability of them choosing Purrfect, and what does this mean with respect to the claim made by the other manufacturer?

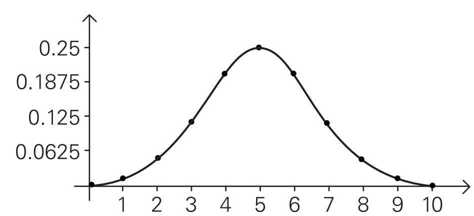
In their solution attempts, students could use the binomial model to calculate the probability that exactly 8 of 10 chose Purrfect:

$$\binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = 45\left(\frac{1}{256}\right)\left(\frac{1}{4}\right) = \frac{45}{1024} \doteq 0.0439$$

If R is the number choosing Purrfect, then the full probability distribution would be:

r	0	1	2	3	4	5	6	7	8	9	10
$P(R=r)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

The graph of this distribution looks quite normal, since the probability is 0.5 that cats will choose Purrfect over the other food choice.



From the graph the result “8 or more out of 10” is likely to occur in only about 5% of all samples of 10 cats.
Based on previous study 5% is not very likely and suggests that the assumption made by the rival manufacturers is probably wrong. It appears likely that more than 50% of cats would indeed choose Purrfect.

The probability of 8 or more of the cats choosing Purrfect can be calculated using:

$$P(8 \text{ or more}) = \frac{45}{1024}+\frac{10}{1024}+\frac{1}{1024}=\frac{56}{1024} \doteq 0.055.$$

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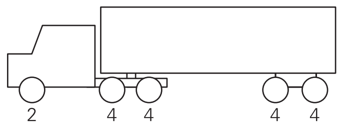
Probability

Worthwhile Tasks for Instruction and/or Assessment

G11(111)/G12(111)

Performance



1) **Eighteen-Wheeler Problem:** Large tractor-trailer trucks usually have 18 tires. Suppose that the probability of any one tire blowing out on a cross-country trip is 0.03.
   Ask students the following:
   a) What is the probability that any one tire does not blow out?
   b) What is the probability that
      i) none of the 18 tires blows out?
      ii) exactly one tire blows out?
      iii) exactly two tires blow out?
      iv) more than two tires blow out?
   c) If the trucker wants to have a 95% probability of making the trip without a blowout, what must the reliability of each tire be? That is, what is the probability that any one tire will blow out?

2) Sally claims that she can predict which way a coin will land, either heads or tails. Tommy throws the coin eight times and Sally gets it right six times. Ask students to calculate, on the basis of a binomial model, the probability of
   a) getting six coin tosses correct out of eight
   b) getting six or more coin tosses correct out of eight
   c) Ask students if they think the result supports her claim? Explain your answer.

6) A blind taste test is organized to see if people can tell the difference between two different brands of orange juice. They have 10 “tastes”. After each taste they have to say whether it is juice A or juice B. Ask students how often they would expect the participants to get it right before they were reasonably convinced that they could actually tell the difference.

7) A list of people eligible for jury duty contains about 40% women. A judge is responsible for selecting six jurors from this list.
   a) If the judge’s selection is made at random, what is the probability that three of the six jurors will be women?
   b) Prepare a probability distribution table and graph for the number of women among the six jurors.
   c) The judge’s selection includes only one woman. Ask students if they think this is sufficient reason to suspect the judge of discrimination? Explain.

Suggested Resources

Unit 4
Circle Geometry
(35 Hours)

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E4 **apply properties of circles**

E5 **apply inductive reasoning to make conjectures in geometric situations**

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

E7 **investigate and make and prove conjectures associated with chord properties of circles**

E8 **investigate and make and prove conjectures associated with angle relationships in circles**

E9 **investigate and make and prove conjectures associated with tangent properties of circles**

E12 **demonstrate an understanding of the concept of converse**

Elaboration—Instructional Strategies/Suggestions

E4 Geometry is a rich field of mathematical study. The world around us is inherently geometric, and hummankind's creations most often reflect geometric principles. The concrete and visual nature of geometry resonates with certain learning styles, and geometry's pervasiveness in our environment facilitates connecting the study of geometry to meaningful, real-world situations. This is as true for circle geometry, the focus of this unit, as for geometry in general. Whether determining the correct location for handles on a bucket, finding the centre of a circle in an irrigation project, or determining the length of a tangent to the earth from an orbiting satellite, properties of circles (and lines, line segments, and/or angles associated with them) come into play.

E11/E5 Geometric figures such as segments, lines, angles, polygons, circles, and planes are each sets of points that are subsets of the universal set called space. In synthetic (Euclidean) geometry, these geometric figures can be drawn anywhere on a plane in space; in analytical (coordinate) geometry, a reference system is added, and important points on the figures are assigned coordinates. Using transformations, these figures—with or without coordinates—can be moved in space by following specific rules. In all perspectives, students seek to discover patterns among figures or within a fixed figure.

Students need many opportunities to explore geometric situations, look for common elements (or patterns) in them, and make appropriate conjectures. They also need to reach an understanding that, while this inductive process of observing multiple cases and conjecturing seems to imply the truth of a relationship, deductive reasoning is required to establish the truth of any conjecture in general. As part of this process, students should also realize that measurements with tools i) are not accurate and ii) deal only with specific cases and are, therefore, not adequate as proofs.

Students should be exposed to a variety of modes of proof, with the understanding that a logical argument can take many different forms. This implies that students should experience Euclidean, coordinate, and transformational approaches and develop an appreciation that each can be advantageous in certain situations. This unit provides the opportunity for students to develop some proficiency with all three modes of proof. If time should be a factor, however, it is sufficient for students to focus on analytical arguments and either Euclidean or transformational.

E7/E8/E9/E5/E11/E12 In particular, contexts will be explored, and theorems conjectured, proven and applied, with respect to chord properties in circles, inscribed and central angle relationships, and tangents to circles. The treatment of these circle topics is not intended to be exhaustive, but is determined to a significant extent by the contexts examined. It should also be noted that the concept of the converse of a theorem will also be explored in relation to some of the theorems developed.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E5/E7

Performance

1) Activity:
   a) Begin with a circle of any size (given to the student).
      – Fold the circle in half—make a crease.
      – Open up the circle and fold in half differently.
      – Open up the circle and investigate the intersection point—compare with your classmates—make a conjecture.
   b) Begin with a circle of any size (given to the student).
      – Make a fold anywhere on the circle, make a crease.
      – Repeat the first step with a second fold and crease.
      – Mark the first fold AB at its end point, the second CD.
      – Fold A onto B, make a crease.
      – Open the circle, fold C onto D, make a crease.
      – Investigate the intersection of the last two creases—compare with your classmates—make a conjecture.
   c) Begin with a circle (make your own) and mark the centre point.
      – By folding create five chords (creases) of different lengths.
      – Fold one end of each chord onto itself, make a crease.
      – Investigate the lengths of these creases from the centre of the circle to the chord—compare with your classmates—make a conjecture.
   d) Begin with any circle (make your own) and mark the centre point.
      – Make fivefolds creating 5 chords all of equal length (fold into the centre).
      – Investigate the distance each is from the centre—compare with your classmates—make a conjecture.

E4/E5/E9

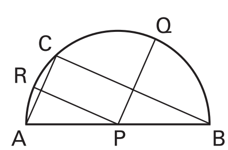
Performance

   e) Begin with a circle on a rectangular sheet of paper and mark the centre.
      – Make a fold at 3 different points on the circumference to produce creases that touch the circle at only those three points.
      – Join the points to the centre and investigate the angles formed between the radii and the tangents—make a conjecture.
      – Place a coordinate system over this situation, or transfer it to a coordinate system, find the slopes of the radius and the tangent to it, compare with your classmates, and make a conjecture.

E4/E11/E8

Performance

2) If P is the centre of semicircle $\overset{\frown}{AB}$, PR bisects $\overset{\frown}{AC}$ and PQ bisects $\overset{\frown}{BC}$, prove PR is perpendicular to PQ.



E12/E11

Performance

3) State and prove the converse of:
   If a point on the hypotenuse of a right triangle is equidistant to all three vertices, then it is the midpoint of the hypotenuse.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E5 **apply inductive reasoning to make conjectures in geometric situations**

E7 **investigate and make and prove conjectures associated with chord properties of circles**

E12 **demonstrate an understanding of the concept of converse**

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

E4 **apply properties of circles**

Elaboration—Instructional Strategies/Suggestions

E5/E7 Students need to begin their study of circles by exploring patterns and making and verifying conjectures. They might begin their exploration with an activity like the following:

❐ *Activity*

- On a blank sheet of paper (or using technology) place any two points *P* and *Q*. Construct a circle that passes through *P* and *Q* such that *PQ* is not the diameter and explain how you located the centre (C). What type of triangle must *PQC* be? Explain.
- Construct three more different circles that pass through *P* and *Q*. Name their centres *D*, *E*, and *F*.
  - Fold *P* onto *Q*, making a crease to indicate the fold line.
  - What do you notice about the points *C*, *D*, *E*, and *F*?
  - Name the point where the crease intersects *P*, *Q*, as *M*. Is *M* the midpoint of *PQ*? Justify your answer.
  - Is $PQ \perp$ to the fold line? How do you know?
  - Make a conjecture. Verify your conjecture using proof.
  - Take any point A on the fold line, join it to *P* and *Q*. Make a conjecture. Verify your conjecture using proof.

E5/E7/E12 While exploring patterns (as in the previous activity), students might use paper-folding techniques and/or measurement tools like rulers, dividers, compasses, and protractors. In so doing, they will be using both transformational and Euclidean techniques. They will also be using inductive reasoning to make conjectures such as

- any point that is equidistant from two points on a circle must be on the perpendicular bisector of the chord joining those two points,

or its converse:

- any point that is on the perpendicular bisector of a chord of a circle must be equidistant from the end points of that chord.

E11/E4 Moving from conjecturing to proving conjectures in general (e.g., proving theorems), and applying new theorems to calculate or prove other geometric facts, may be a large step for many students. Teachers will need to model the thinking processes necessary to generate proofs. As well, it may be necessary to spend time reaquainting students with the geometric properties and theorems with which they are already familiar (e.g., congruent triangles, angle sum of a triangle, vertically opposite angles, parallel line theorems). See p. 118 for a further elaboration.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E5/E4/E7/E11

Performance

1) Construct two circles with radius *r* so that each circle passes through the centre of the other circle. Label the centres P and Q, and construct the segment PQ. The two circles intersect at A and B.
   a) What is the relationship between the segments AB and PQ? Explain your thinking.
   b) Prove your conjecture in (a).

2) Construct a large circle and two non-parallel congruent chords that are not diameters.
   a) Compare their distance to the centre of the circle.
   b) Write your findings in (a) as a conjecture.
   c) Test your conjecture on other circles.
   d) Prove your conjecture.

E12

Performance

3) a) Restate the conjecture you made in question 2 above in an "if ... then ..." form.
   b) State the converse of this conjecture.
   c) Is the converse true? Explain.

4) State a theorem related to geometry whose inverse is not true.

E4

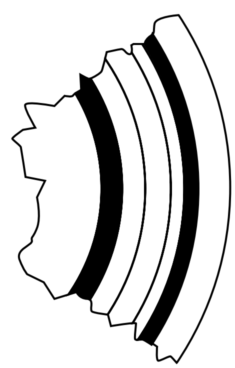
Performance

5) Use a circular object to trace a circle onto your paper. Without using a compass, locate the centre of the circle.

E4/E7

Performance

6) A piece of circular plate was recently dug up on an island in the Mediterranean. The discoverer of the plate wishes to calculate the diameter of the original plate. Describe how he could do this.



Suggested Resources

Hirsch, Christian R. ed. Curriculum and Evaluation Standards for School Matheamtics. Addenda Series. A Cone Curriculum. Reston, VA: NCTM, 1992.

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E12 **demonstrate an understanding of the concept of converse**

E7 **investigate and make and prove conjectures associated with chord properties of circles**

E8 **investigate and make and prove conjectures associated with angle relationships in circles**

Elaboration—Instructional Strategies/Suggestions

E12/E7 Once students begin to articulate conjectures and try to establish them as theorems, it would be appropriate to introduce them to the concept of converse. Essentially, a converse is an opposite, or something reversed in order or action. Consequently, the converse of a conjecture or theorem will be reversed in order and/or action with respect to the original.

Students may find, when they examine the conjectures that they made in their initial explorations, that some are indeed converses of others. They may find this easiest to do if they adopt the conventional, conditional form for statements of conjectures/theorems, e.g., the "if ..., then ..." statement. For example, in this format the statements from p. 114 become

- If a point is equidistant from two points on a circle, then it lies on the perpendicular bisector of the chord joining those two points
- If a point lies on the perpendicular bisector of a chord of a circle, then it is equidistant from the end points of that chord.

E12/E8/E7 In general, students should understand that the converse of any "if *p*, then *q*,"statement is "if *q*, then *p*." As well, it is critical that students understand that the truth of any theorem does not necessarily imply the truth of its converse. In some cases a converse is also true, in other cases it is not, so students should always test the truth of a converse. For example, they might write the converse of "if an angle is inscribed in a semi-circle, then it is a right angle" and see that the converse is not true in general. The discussions arising out of the examination of these types of examples encourage logical thinking.

Students should know that if a statement and its converse are true, it can be stated as an "if and only if"(iff) situation. This gives rise to sufficient conditions: e.g., points lie on the perpendicular bisector of a chord iff they are equidistant from the end points of the chord. Students should realize that it is sufficient to prove a line is a perpendicular bisector of a chord by proving two points on it are equidistant from the endpoints of the chord.

Converse is not a concept that requires extensive attention in its own right. It should be addressed from time to time as it comes up throughout the unit, with a view to students' developing a clear understanding of the concept.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E12/E7

Performance

1) State the converse of each of the following. If you think the converse is true, rewrite using 'if and only if" (iff). If you think it is false, explain why.
   a) If a triangle has two angles of equal measure, then it is isosceles.
   b) If two triangles are congruent, then their corresponding angles are congruent.
   c) A quadrilateral with four axes of symmetry is a square.
   d) Every square has four sides of equal length.
   e) The centre of a circle lies on the perpendicular bisector of a chord of that circle.
   f) A tangent line is perpendicular to the radius of a circle.

2) State and prove the converse of:
   If a point on the hypotenuse of a right triangle is equidistant to all three vertices, then it is the midpoint of the hypotenuse.

3) Prove the converse of this statement:
   If a line passes through the midpoint of a chord on a circle, then it is equidistant from the endpoints of the chord.

4) Make a statement and state its converse so that the original statement is true but the converse is not. State a theorem in geometry that you know is true but whose converse is not.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

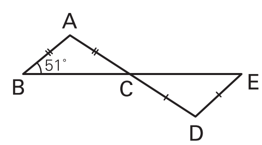
E4 **apply properties of circles**

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

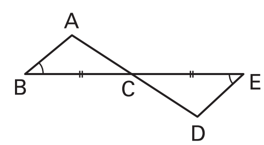
Elaboration—Instructional Strategies/Suggestions

E4/E11 As previously indicated, it may be necessary to refresh students' knowledge of geometric properties and theorems (e.g., congruent triangles) while building strength with respect to writing deductive arguments. This may be accomplished by applying deductive reasoning in situations in which students are asked to apply their knowledge of geometry to i) determine specific angle measures and/or ii) prove geometric facts or theorems in general. Note: This is an opportunity to expose students to both Euclidean and transformational forms of proof.

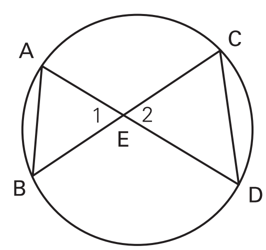
Sometimes students might be given some angle measures and be asked to find and verify that another angle has a certain measure. For example, given the situation to the left, find and verify that the angle *CDE* must be 78°. Students could begin with the given 51° angle and transfer equality to $\angle CED$ using vertically opposite angles and equal angles in an isosceles triangle, then finally get $m\angle CED = 78°$ using the fact that the angles in a triangle add to 180°.



In a slightly more general situation, students could be asked to prove that *AC* = *DC*. Using 180° rotation, centre *C*, students could state that $B \leftrightarrow E$ because *B* – *C* – *E* and *BC* = *CE* then $A \to \overrightarrow{ED}$ because $\angle ABC = \angle CED$, so $A \to D$ because *AB* = *DE* since *C* is the turn centre $AC \to CD$ and $\therefore AC = CD$ because in a rotation the image must equal the object.



When students are presented with arguments that are not valid they should be able to identify the flaw. For example:



Given: *AD* and *BC* intersect at *E*.

Jon argues that *AB* = *CD*. His argument goes like this:

$\angle 1 = \angle 2$ because opposite angles are equal,

AE = *ED*, *CE* = *ED* because of equal radii.

So the triangles are congruent and *AB* = *CD* since they are corresponding parts of congruent triangles. Find the flaw.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E11

Performance

1) Prove that the incentre of a triangle is the centre of the inscribed circle of that triangle.

2) Prove that the circumcentre of a triangle is the centre of the circle circumscribed on that triangle.

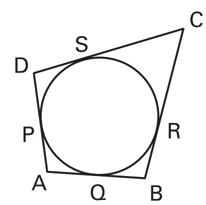
3) Use transformations to prove that the centre of a circle is the intersection of the perpendicular bisectors of two chords.

4) Use transformations to prove that two chords equidistant to the centre have the same length.

5) Use transformations to prove that a tangent to a circle is perpendicular to the radius at the point of tangency.

6) Use congruent triangles to prove that tangent segments to a circle from a point outside the circle are equal in length.

7) If the sides of a quadrilateral ABCD are tangent to the circle, show that AB + DC = AD + BC.



8) Find the flaw(s) in the following proof:

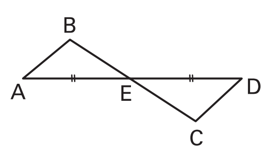
Prove: $\angle B = \angle C$

Proof: for a rotation, centre E

$A \rightarrow D(AE = ED)$

$B \rightarrow C(AB \parallel ED)$

$\therefore \angle B = \angle C$



Journal

9) Describe how the following idea could be proved:
The centre of any circle is the intersection of the perpendicular bisectors of any two non-parallel chords in the circle.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

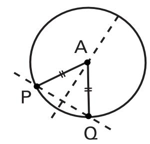
E7 **investigate and make and prove conjectures associated with chord properties of circles**

Elaboration—Instructional Strategies/Suggestions

E11/E7 Once students have been reaquainted with the necessary geometric ideas, they can move on to applying them in the context of properties of circles, beginning with chord properties.

Students should write proofs using deductive arguments and explain that the validity of the arguments are valid. For example, in proving the conjecture

- Any point that is equidistant from two points on a circle must be on the perpendicular bisector of the chord joining those two points



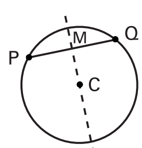
Students might say ... "Since $AP = AQ$ (A is equidistant to two points), then the triangle APQ is isosceles and has one line of symmetry passing through the vertex angle, and $\angle A$ is the vertex angle.

"A property of symmetry says that this line of symmetry must be the perpendicular bisector of the base of an isosceles triangle, thus proving the conjecture."

Alternatively, students might use a Euclidean version of the same proof:

"Join A to M, where M is the midpoint of segment PQ; the triangles would now be congruent since there are three pairs of corresponding sides congruent (SSS). Since the triangles are congruent, then the corresponding angles at M inside the triangles are congruent, and they are supplementary, so each is a right angle (two 90s make 180). Since the angles at M are right angles, then AM is perpendicular to PQ at its midpoint, and the conjecture is proved."

By way of a second example, consider the following:



Given that P and Q are points on a circle with centre C, students could fold so that

$P \to Q$. Since $PC = CQ$ (C is the centre of the circle), C folds onto itself,

$C \to C$, e.g., C must be on the mirror line. Properties of reflection say that the mirror line must be perpendicular to a line joining a point to its image (P and Q). So if the intersection point is named M, then $CM \perp PQ$.

Some students might want to prove $\triangle CMP = \triangle CMQ$ in order to prove $CM \perp PQ$. Their proof may look like this:

Let M be the midpoint of PQ

$\triangle CMP \cong \triangle CMQ$ (SSS)

$\angle CMP \simeq \angle CMQ$ $(\triangle\text{'s} \cong)$

$\therefore m\angle CMP = 90° = m\angle MCQ$ (a congruent and supplementary)

$\therefore CM \perp PQ$, and CM bisects PQ (M is the midpoint)

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/E7

Performance

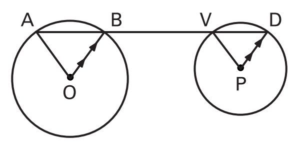
1) Mary said to Beth: "I can draw a segment inside the circle that is equal in length to one half the diameter without measuring it." Beth was impressed and insisted that Mary show her work. Mary told Beth to draw any circle, with diameter AB then to draw two more chords to make the triangle ABC. She told Beth that, if she connects the two midpoints of the two chords just drawn, the segment joining them would be one half the length of the diameter. Explain how you know that the segment joining the midpoints would be one-half the diameter.

2) Lou, who was listening to and watching Mary and Beth, was startled because now, he exclaimed, he now knew how to prove that the segments joining the midpoints of any quadrilateral form a parallelogram. How can Lou do this?

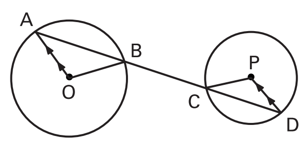
3) Draw a circle. Draw two chords of the circle of unequal length. Which is closer to the centre of the circle? Prove it.

4) In designing various logos that use circles, Frank wanted to make sure that what seemed to look correct really was correct. Help him prove the following relationships:

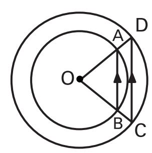
a) Given: $OB \parallel PD$
   O and P are centres of circles
   Prove: $\angle OAB = \angle PCD$



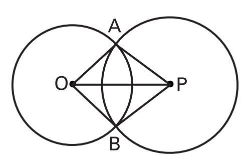
b) Given: $OA \parallel PD$
   O and P are centres of circles
   Prove: $\angle OAB = \angle PCD$



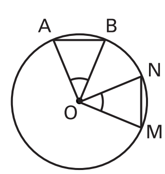
c) Given: $AB \parallel CD$
   O is the centre of both circles
   Prove: $\angle OAB = \angle C$



5) O and P are the centres of the circles.
   a) Why is the figure $OBPA$ a kite?
   b) Explain why $AB \perp OP$
   c) What other conclusions are valid?



6) Given that O is the centre of the circle, and the two central angles are congruent, prove $AB = MN$



Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

D1 **develop and apply formulas for distance and midpoint**

E5 **apply inductive reasoning to make conjectures in geometric situations**

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

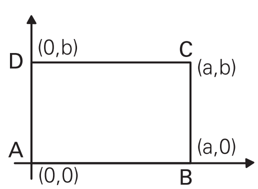
Elaboration—Instructional Strategies/Suggestions

D1 It is common for students to use measurement to attempt to prove conjectures. However, for generalization, they need to develop a way of generating length or distance algebraically. They might begin by contrasting the ease of determining the lengths of horizontal and vertical lines (on a coordinate system) with the difficulty of finding the lengths of oblique lines. (For example, they might compare finding the distance of the point (3, 0) from the origin as compared to finding the distance of (3, 5) from the origin.) The connection between the Pythagorean Theorem and the distance formula needs to be emphasized and understood, so that students can connect this new formula to previous knowledge. In the same way, the calculation of the midpoint can be connected to averaging.

Distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

E5/E11/D1 In circles, there are several opportunities for line segments to be perpendicular. Opportunity should be provided for students to conjecture that when the slopes of two segments are negative reciprocals the two segments are perpendicular. Sometimes in coordinate proof, students can benefit from knowing that the product of the slope of a line and its reciprocal is –1. This may sometimes be easier for students to see than that the fact that they are negative reciprocals of each other. Once students have made such a conjecture, it needs to be proven. Students could then apply (parallel properties) to prove that, given four points, certain segments are parallel and others are perpendicular, and that some quadrilaterals are parallelograms and some are rectangles or rhombi.

The coordinate system can also be used to help generalize properties. For example, in proving that the diagonals of a rectangle are congruent, students would assign variable coordinates to the four vertices, then express the length of the diagonals *DB* and *AC* as



$$DB = \sqrt{(a-0)^2 + (0-b)^2} \qquad AC = \sqrt{(a-0)^2 + (b-0)^2}$$

$$= \sqrt{a^2 + b^2} \qquad = \sqrt{a^2 + b^2}$$

and conclude that their lengths must be equal.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

D1/E11

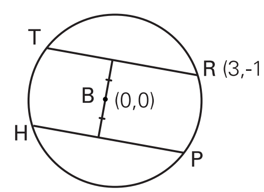
Performance

1) A sprinkler head is positioned on the infield so that the spray of the water soaks the entire field that lies within the 10-metre radius of the head. Assume that a Cartesian coordinate system has been placed over the field and the sprinkler head is at the coordinate *S* (3, –1). Determine whether or not the new trees at the following locations will get wet or not?
   a) *A* (11, 4) b) *B* (–3, –9) c) *C* (12, –6) d) *D* (–4, –8)
2) Another sprinkler head is positioned at *P* (–2, 6). A tree is on the circumference at (5.5, 7). What is the area of the ground that gets wet?
3) These trees *P* (3, 6), *Q* (6, -1), and *R* (-1, -4) lie on the circumference of the water circle. Prove that *C* (1, 1) is the location of the sprinkler head.
4) Determine the coordinates on the field so that the planting of trees *A*, *B*, and *C* lies on the same line and between trees *M* (–17, –8) and *N* (47, 20) so that segment *MN* is divided into four equal parts.

D1/E5/E11

Performance

5) Use Geo-strips to construct a parallelogram. Investigate the many parallelograms and the lengths of their diagonals that can be formed by repositioning the Geo-strips.
   a) Prove that the diagonals of a parallelogram are not congruent.
   b) Prove that the diagonals of a rhombus are congruent.
6) Given C_1: $2y - 6x + 2 = 0$ C_2: $5x + 5y - 15 = 0$ the equations for two chords of a circle, prove that the intersection of these chords is the centre of the circle that contains a diameter that runs from *A* (–4, 2) to *B* (6, 2).
7) Given that the radius bisects the central angle in a circle, prove that it bisects the arc subtending the central angle.
8) Given the diagram, ask students to make up a question to find the coordinates for the point *P*.



Journal

9) Describe how the procedure for calculating the distance between two points, given their coordinates, is similar to the procedure for calculating the slope of the line that joins the two points. Describe how it is different.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

D1 develop and apply formulas for distance and midpoint

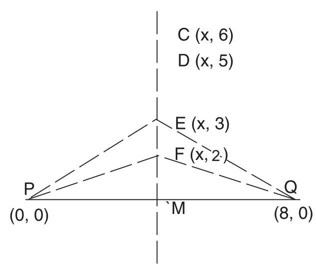
Elaboration—Instructional Strategies/Suggestions

E11/D1 Students might revisit statements that they have already provided using Euclidean or transformational proofs. For example, students have already shown that points equidistant to the two points *P* and *Q* (see p. 120) are on a line ⊥ to *PQ* and passing through the midpoint. Using coordinate geometry to assign coordinates to *P* and *Q* ask students to find the coordinates for the centre points *C*, *D*, *E*, and *F*, knowing that radii of circles are equal, then find the slope of $\overrightarrow{PQ}$ and $\overrightarrow{CD}$. This should lead to a conjecture that there is a perpendicular relationship of *PQ* and *CM* through midpoint *M*.

$$PF = \sqrt{(x-0)^2+(2-0)^2} = \sqrt{x^2+4}$$

$$QF = \sqrt{(x-8)^2+(2-0)^2} = \sqrt{x^2-16x+64+4} = \sqrt{x^2-16x+68}$$

Assume F is the centre of the circle:



$$\sqrt{x^2+4} = \sqrt{x^2-16x+68}$$

$$\therefore x = 4, \text{ and F's coordinates are } (4,2)$$

$$\text{slope of PQ}=\frac{\text{rise}}{\text{run}} = \frac{0}{8} = 0$$

$$\text{slope of FC}=\frac{4}{0} = \infty$$

$$\therefore \text{FC} \perp \text{PQ}$$

They will re-examine these statements and attempt to prove them using coordinate arguments.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/D1

Performance

1) A triangle has vertices R (17, 16), S (1, 4), and T (7, –4).
   a) Prove that the triangle is right-angled.
   b) M is the midpoint of RT. Prove that a circle with centre M and passing through R, also passes through T and S.

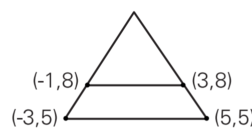
2) Find the equation for the set of all points that are equidistant from (6, 2) and (1, –5).

3) Given a right triangle with vertices A (1, 4) and B (9, 3). The third vertex C is on the x-axis. If side AB is the hypotenuse, find the coordinates of C. Is there more than one answer? Explain.

4) Ralph decided to use coordinate geometry to help him solve his problem. He located the middle of a beam at M (2, 3). He was able to locate approximate positions for the endpoints P and Q. He knew the x-value for the point P (–4, y), and the y-value for the point Q (x, -2). Find the values for x and y.

5) A ceiling support beam is constructed from several congruent isosceles triangles. When the midpoints of one of the triangles are joined will the new triangle also be isosceles?

6) Create a real-life problem using the diagram on the right.



7) Using transformations, prove that the midpoint, M, of a line segment PQ with endpoints P (a, b), and Q (c, d) is $M(\frac{a+c}{2}, \frac{b+d}{2})$.

8) Two designers are using a Cartesian plane to design a large plus sign to hang in the operations room. The plus sign is made using two perpendicular lines whose equations are almost completely determined: $x - 2y = -8$, and $kx - y = -3(x + 1)$. Help them determine the value for k. Is there more than one value for k? Explain.

9) Determine the ratio of the sum of the lengths of the three altitudes to the perimeter of the triangle whose vertices are P (0, 0), Q (4, 3), and R (–2, 5). Do you think that this ratio is the same for any triangle? Investigate.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

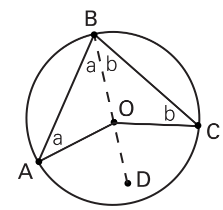
E8 **investigate and make and prove conjectures associated with angle relationships in circles**

E9 **investigate and make and prove conjectures associated with tangent properties of circles**

Elaboration—Instructional Strategies/Suggestions

E11/E8/E9 Students will continue to apply their knowledge of axioms and algebra to construct logical arguments with respect to other properties of circles, in particular angles in circles and tangents to circles. For example, to prove that the inscribed angle is half the central angle, students might use synthetic geometry:

Draw the line through *BO*. In triangle *AOB*: *OB* = *OA* because of equal radii $m\angle A = m\angle OBA$ = "*a*" because triangle *AOB* is isosceles. Thus, $m\angle AOD = 2a$ since the exterior angle of a triangle is the sum of the two remote interior angles.



Similarly, $m\angle DOC = 2b$

So, $m\angle AOC = 2a + 2b$

$= 2(a + b)$

$m\angle ABC = a + b$

$\therefore m\angle AOC = 2m\angle ABC$

Students could be asked to make the logical argument that an angle in a semicircle is 90°. Some students might say that an inscribed angle is half of the central angle and since the central angle is 180°, the inscribed angle must be 90°. Other students might construct two isosceles triangles as in the above diagram and show that when $2a + 2b = 180°$ results the equation $a + b = 90°$ is logical for $\angle ABC$. The "proofs" need not be long and involved. What is important is that they are based on agreed-upon axioms within the geometry system (synthetic, transformational, coordinate) being used.

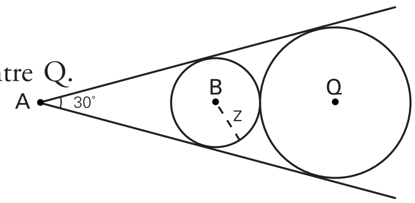
Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/E8/E9

Performance

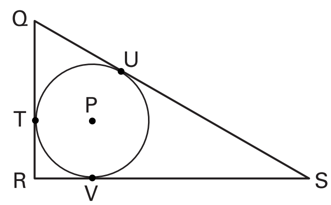
1) Find the diameter for the circle with centre Q.



2) Given: $\angle R$ is a right angle T, U, and V are points of tangency with the circle centre P.

   Prove: $r = \frac{1}{2}(QR + RS - QS)$

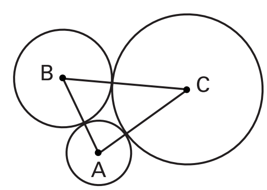
   when r is the radius of circle P.



3) Prove:
   a) The central angle of a circle is twice the measure of the inscribed angle on the same arc.
   b) The angle inscribed in a semi-circle is a right angle.

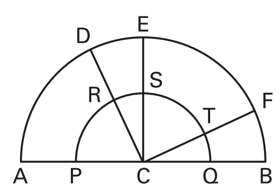
4) Two circles with centres P and Q are tangent at S. Prove that P, S, and Q are collinear points.

5) In the diagram, each circle is tangent to the other two. If AB = 10 cm, AC = 14 cm, and BC = 18 cm, find the radius of each circle.



6) $\widehat{AB}$ is a semicircle with centre C, $\widehat{PQ}$ is the concentric with $\widehat{AB}$, $EC \perp AB$, and $CD \perp CF$.

   Prove $m\widehat{AD} + m\widehat{QT} = m\widehat{EF} + m\widehat{RS}$.



7) Construct two non-parallel, non-congruent chords on a circle of any radius. Connect the endpoints of the chords with segments so that the segments intersect. Measure the four angles formed at the circumference
   a) Make a conjecture about two angles subtended by the same arc.
   b) Construct a central angle and compare the measures of a central angle with the inscribed angle subtended by the same arc. Make a conjecture and check it with other central angles in the diagram.
   c) Prove the conjecture in (b), then use that proof to prove the conjecture in (a).

Suggested Resources

McKillop, David W. et al. *Pre- Calculus Mathematics One.* Toronto: Nelson, 1992

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E4 **apply properties of circles**

E15 **solve problems involving the equations and characteristics of circles and ellipses**

Elaboration—Instructional Strategies/Suggestions

E4/E15 Many problems can be solved with the drawing of a circle or circles. Students will explore how technology (especially using the Draw Mode on graphing calculators) can be used to construct circles.

All students should be able to attempt to solve problems involving circles such as the following:

The planners of the arena want to have a smaller rectangular grassed area on which games like croquet, badminton, tennis, dodge ball, etc., might be played. Suppose the field might have dimensions 75 m by 50 m. The groundskeeper for the field wants to position three sprinklers. Each sprinkler throws water out over a semi-circular path. He wants to position the three water heads on three of the four boundaries of the field so that every place in the field will be exposed to water. He places one at the centre of the front 50 m wall, and one on each of the 75-m walls, 55 m back from the front wall. Will sprinkler heads that cover semi-circles up to 30 m be adequate, or will he have to purchase the expensive 35-m size or even more expensive 40-m size?

Students might begin to explore this problem by drawing circles (or by using their graphing calculators). If using calculators, they would set their window to represent the field dimensions, then draw circles using the three sprinkler head locations as centres. In the Draw menu, select 9: circle (0, 25, 30), then push Enter. This will draw a circle centre (0, 25) radius 30 m.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E15

Performance

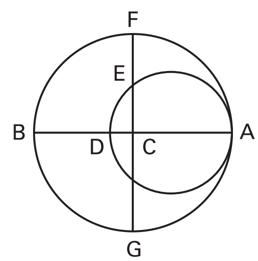
1) How many circular pipes, each with an inside diameter of 10 cm, will carry the same amount of water as a pipe with an inside diameter of 60 cm?

2) By how much does the radius of a circle increase when the circumference is increased from 20 cm to 25 cm?

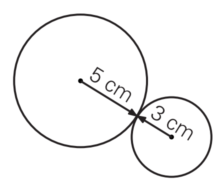
3) A 16-cm chord is 15 cm from the centre of a circle. What is the radius of this circle?

4) An 18-cm chord is perpendicular to the radius of a circle. The distance from the intersection of the chord and the radius to the outer end of the radius is 3 cm. What is the length of the radius?

5) Two circles are internally tangent at *A*. *C* is the centre of the larger circle. BA is perpendicular to *CF*, *EF* = 5 cm and *BD* = 9 cm. What is the length of the diameter of the smaller circle?



6) Two gears have radii 5 cm and 3 cm. How many times must the smaller gear be turned in order for the arrows on each gear to align again?



7) For the past several years on opening day of lobster season, the weather and other circumstances have caused life-threatening incidents. This year the air sea rescue helicopters are to be placed so that every point in the 90-km square region can be reached within 20 minutes. In 20 minutes these helicopters can travel up to 36 km. Three plans are proposed.
Assuming that you will use an appropriate window on your calculator:
Plan A
- Five helicopters
- One in the middle
- One at each corner

Plan B
- Four helicopters at (20, 20), (20, 70), (70, 20), (70, 70)

Plan C
- Four helicopters at (25, 25), (25, 65), (65, 25), (65, 65)

How can you use the circle-drawing feature of your graphing calculator to evaluate these three plans? Do it. Which is best? Which is worst? Why? Can you create a better plan?

Suggested Resources

McKillop, David W. et al. *Pre- Calculus Mathematics One*. Toronto: Nelson, 1992

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E15 solve problems involving the equations and characteristics of circles and ellipses

E4 apply properties of circles

D1 develop and apply formulas for distance and midpoint

Elaboration—Instructional Strategies/Suggestions

E15/E4/D1 In exploring the sprinkler system described in the elaboration for E15, p. 128, it would be helpful to have other ways to decide whether or not given points are within the circular boundary of the spray given off by the sprinkler heads. For example, have students determine if the following points are within the 10 unit boundary, on the boundary, or outside the boundary of the circular spray when the sprinkler head is given the coordinates (0, 0):

a) (9, 4) b) (8, 5) c) (8, 6) d) (9, 6)

Some students might do this using the graphing calculator approach, while others may draw diagrams with compasses or use the distance formula.

It might help to have students focus on exactly what it means to have a circle with radius 10 units. They should come up with a description of such a circle that generalizes to "A circle is the set of all those points in a plane that are a given distance (the radius) from a given point (the centre).

Using the distance formula, students should determine if specific points are within the boundaries of the circle with radius 10. Ask students what equation could be written using coordinate variables *x* and *y*, so that the graph of the equation is a circle.

Students should test whether the point P (8, 6) lies on the circle using the distance formula:

$$OP = \sqrt{(8-0)^2 + (6-0)^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$$

When (*x*, *y*) is used to replace (8,6), the equation describes any point on the circle.

$$OP_1 = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

They know the radius (OP_1) is 10, so:

$$\sqrt{x^2 + y^2} = 10 \text{ or } x^2 + y^2 = 10^2$$

The equation of a circle with radius 10 can be expressed as $x^2 + y^2 = 10^2$. Emphasize with the students, that this equation produces a circle, Students can now conjecture that

1) If the coordinates of a point satisfy the equation, then the point is on the circle.
2) If a point is on the circle, then its coordinates satisfy the equation.

This is another opportunity to use the term converse.

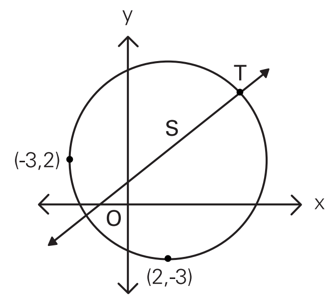
Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E15/E4/D1

Pencil and Paper

1) A ranger station, S, is located on the line $2x - 3y = -2$. The area of sight from the station is bounded by the circle.
   a) Find the equation of the boundary.
   b) If a second station is located at T, find its coordinates.



2) A circle is defined by $x^2 + y^2 = 10$.
   a) Show that AB is a chord of the circle where $A\ (1, -3)$ and $B\ (-3, 1)$.
   b) Find the equation of the right bisector of AB.
   c) Show that the right bisector of AB passes through the centre of the circle.

3) The equation of a circle is given by $x^2 + y^2 - 4x + my - 18 = 0$. If $A\ (7, 3)$ is a point on the circle, find the value of m.

4) Prove that the line $x - 3y + 24 = 0$ passes through the centre of the circle given by $x^2 + y^2 + 12x - 12y + 27 = 0$.

5) a) Graph the region defined by the inequalities $x^2 + y^2 - 2x + 4y - 5 \le 0$ and $x + y - 1 \ge 0$.
   b) Determine the area of the region defined in (a).

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E13 **analyse and translate between symbolic, graphical, and written representations of circles and ellipses**

E3 **write the equations of circles and ellipses in transformational form and as mapping rules to visualize and sketch graphs**

E16 **demonstrate the transformational relationship between the circle and the ellipse**

E14 **translate between different forms of the equations of circles and ellipses**

E15 **solve problems involving the equations and characteristics of circles and ellipses**

E7 **investigate and make and prove conjectures associated with chord properties of circles**

E11 **write proofs using various axiomatic systems and assess the validity of deductive arguments**

Elaboration—Instructional Strategies/Suggestions

E13/E3/E16 Reconsider the sprinkler system problem. Suppose the sprinkler head is moved to a new location *Q* (3, –1). Keep the radius 10 and find the equation of the circle. From their earlier work:

$$RQ = \sqrt{(x-3)^2 + (y+1)^2}$$
$$\Rightarrow (x-3)^2 + (y+1)^2 = 10^2$$

From this centre-radius form of the equation, students can see the coordinates for the centre of the circle. Just as they would see the vertex for the parabola in a quadratic equation. This circle equation could be expressed in transformational form as $\left[\frac{1}{10}(x-3)\right]^2 + \left[\frac{1}{10}(y+1)\right]^2$ where the stretch factors give the radius of the circle. As a mapping rule the image equation can be expressed as $(x.y) \rightarrow (10x+3, 10y-1)$.

If the water pressure was turned up and the sprinkler head could now throw water 12 units, the equation would become $\left[\frac{1}{12}(x-3)\right]^2 + \left[\frac{1}{12}(y+1)\right]^2 = 1$.

Extending the understanding of the equation of a circle to that of the equation of an ellipse is a simple matter of understanding that the equation of the ellipse is generated from the equation of a circle by having different vertical and horizontal stretches. Its equation, in transformational form, would be $\left[\frac{1}{m}(x-h)\right]^2 + \left[\frac{1}{n}(y-v)\right]^2 = 1$.

E14 Completing the square is used to manipulate of equations of circles and ellipses into transformational form, or in centre—radius form. For example, when students are asked to graph $x^2 + 2x + y^2 - 4y = 12$, they would write the following:

$$x^2 + 2x + y^2 - 4y + \_\_\_ = 20$$
$$x^2 + 2x + \_\_\_ + y^2 - 4y + 4 = 20 + 1 + 4$$
$$(x+1)^2 + (y-2)^2 = 25 \text{ (this is the centre-radius form)}$$
$$\left[\tfrac{1}{5}(x+1)\right]^2 + \left[\tfrac{1}{5}(y-2)\right]^2 = 1 \text{ (this is the transformational form)}$$

From this form they can describe the transformations of $x^2 + y^2 = 1$: "The centre (0, 0) is translated left 1, and up 2, and the radius is 5 units."

E15/E7/E11 Students should solve problems using the equations for circles and ellipses. For example: A circle is defined by $x^2 + y^2 - 10x - 10y + 25 = 0$. Show that the line joining *A* (2, 1) to *B* (5, 0) is a chord of the circle, and that the right bisector of *AB* passes through the centre of the circle.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E3/E16

Pencil and Paper

1) Find the equation of the image of $x^2 + y^2 = 1$ under each mapping. Write the equation in standard form.

   a) $(x, y) \rightarrow (3x - 2, \frac{1}{2}y + 1)$ b) $(x, y) \rightarrow (5x, y - 5)$

   c) $(x, y) \rightarrow (x + 3, y - 2)$ d) $(x, y) \rightarrow (\sqrt{3}x + 5, \sqrt{3}y)$

   e) a vertical stretch of $\frac{1}{2}$, a horizontal stretch of 3, centre (–2, 5)

   f) $(x, y) \rightarrow \left(\frac{1}{3}x - 2, \frac{1}{3}y - 1\right)$

2) What is the centre for the circle or ellipse defined by each equation above? If any of the above are circles, state the radius. If any are ellipses, state the length of the major and minor axis.

E13/E3/E14

Pencil and Paper

3) Express each equation in transformational form state the transformation of $x^2 + y^2 = 1$, and the mapping rules then sketch the graph.

   a) $x^2 + y^2 - 8x - 9 = 0$ b) $x^2 + y^2 - 10x - 6y - 2 = 0$

   c) $2x^2 + 2y^2 + 5y = 0$ d) $2x^2 + 2y^2 - 4x + 6y - 2 = 0$

   e) $x^2 + y^2 - 8x + 6y - 11 = 0$

E15

Pencil and Paper

4) The dome of an arena is elliptical in shape. If the height of the dome is 28 m, and it has a span of 75 m, find a possible equation for this ellipse.

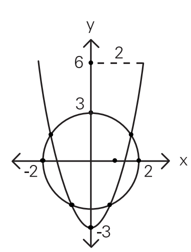
5) An ellipse is given by the equation $25x^2 + 4y^2 + 100x - 16y + 16 = 0$.
   a) What are the coordinates of its centre?
   b) Define the mapping applied to the circle $x^2 + y^2 = 1$ to obtain this ellipse.

6) The points $P\left(-1, \frac{8\sqrt{2}}{3}\right)$ and $Q\left(2, \frac{-4\sqrt{5}}{3}\right)$ are on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the values of a^2 and b^2.

E15/E7/E11

Pencil and Paper

7) Given the circle and ellipse as in the graph
   a) Determine the equation for both the circle and the ellipse.
   b) Determine the length of the two chords.
   c) Determine the larger ratio:
      i) the length of the longer chord to the centre of the ellipse
      ii) the length of the shorter chord to the centre of the circle



Suggested Resources

Circle Geometry

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Outcomes

SCO: In this course, students will be expected to

C20 (111) **represent circles using parametric equations**

C36 (111) **demonstrate an understanding of the relationship between angle rotation and the coordinates of a rotating point**

Elaboration—Instructional Strategies/Suggestions

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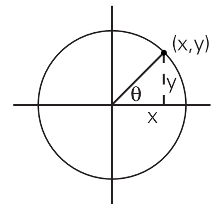
* **C20(111)** Students should investigate how technology "draws" circles. For example, a calculator calculates coordinates for points and so turns numbers into shapes on the screen.

C36(111) Students should understand that if the cursor traced a circle on the screen, it would report coordinates (x, y) along the circle. To simulate this, students should draw a unit circle (radius 1) with centre (0, 0). Try $\theta = 30°$, where the radius meets the circle drop a perpendicular to the x-axis and calculate the value of x, or the length from the origin to the intersection point $\cos\theta = \frac{x}{radius} \Rightarrow \cos 30° = \frac{x}{1} \Rightarrow x = \cos 30°$.



Similarly, $\sin 30° \Rightarrow \frac{y}{1} \Rightarrow y = \sin 30°$.

If $\theta = 135°$, the coordinate where the radius meets the circle is (cos 135°, sin 135°). Likewise, if $\theta = 238°$, the calculator would plot the point (cos 238°, sin 238°) or approximately (–.53, –.85).

Also students should then be able to find the angle of rotation (θ) given a coordinate on the unit circle such as (.34, .94). They should approximate θ to be about 70°.

C20(Z) The angle θ is called the parameter of x=cos θ, y = sin θ. It is a variable used to describe another variable. Equations that contain parameters are called parametric equations. So, if students wanted to draw a circle using parametric equations, they would assign θ the values 0° to 360° and graph x=cos θ, y = sin θ.

On the TI-83 calculator, go to parametric mode and in the window, set $T = 0° \rightarrow 360°$, T-step at 5, x-values from –3 to 3, and y-values from –2 to 2. Then at "y =," students would enter x_{1T}= cos (T) and y_{1T} = sin (T). Push Enter to graph a circle with radius one.

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Circle Geometry

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Worthwhile Tasks for Instruction and/or Assessment

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* **C36(111)**

Paper and Pencil

1) Without using trigonometric ratios, calculate the coordinate of the point P as it rotates through the following degrees on a unit circle, centre (0, 0). Express your answer using exact values.

a) 45°	e) 30°	i) 60°
b) 135°	f) 150°	j) 120°
c) 225°	g) 210°	k) 240°
d) 315°	h) 330°	l) 300°

2) Describe the patterns you see in the above results.
3) Find the same values in (1) above using trigonometric ratios and a calculator.
4) Find the values cos 50° and sin 50°. Explain the value of each in terms of a unit circle.
5) On a unit circle, the coordinates of the image of (1, 0) after a rotation are (–0.6282, 0.7781).
   a) find the angle of rotation correct to two decimal places.
   b) What would be the arc measure from (-1, 0) to (-0.6282, 0.7781)?

C20(111)

Paper and Pencil

6) Using parametric mode, write the equations that will produce a unit circle with centre (0, 0). State your 'window' settings.
7) How would you change your equations in # 6) above to increase the radius, and produce a circle with radius 5 units. State your window settings.

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Suggested Resources

Circle Geometry

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Outcomes

SCO: In this course, students will be expected to

C37 (111) **describe and apply parameter changes within parametric equations of circles**

Elaboration—Instructional Strategies/Suggestions

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* **C37(111)** Students should be encouraged to investigate all the parameters in the list below:

- What happens as T-step is increased? Explain this behaviour. (Fewer points are being calculated, which because of the pixel display has a positive effect on the appearance of the circles up to t-step about 20).
- With x set to –6 to 6, and y to –4 to 4, investigate circles produced by 2 cos T, 2 sin T; 4cos T, 4 sin T, and explain how the "2" and "4" affect the circle (the "2" and "4" are the actual radii of the circles).
- Now explore the graphs of 2 + cos T, 2 + sin T; 4 + cos T, 4 + sin T (the "2" moves the centre to (2, 2), the "4" moves the centre (4, 4), and a small T-step is better. Also, a better window is needed—add two of each of the previous values.
- Explore adding a value to T in the argument: cos (T + 5), sin (T + 5); cos (T +1), sin (T + 1). (Same circle—the "T" value has been incremented, which doesn't affect the size or position of the circle.)
- Explore multiplications of T in the argument: cos 2T, sin 2T; cos 5T, sin 5T. (Again, this produces the same circle—affects T-step—not the position or size of the circle.)

In summary, when the variable T is multiplied by numbers, the radius is affected, but when numbers are added to T, the location of the centre of the circle is transformed.

Students will want to justify their conjectures by making up examples, estimating position and size, then graphing to check.

For example: $3\cos\theta$, $3\sin\theta$, $7 + 3\cos\theta$, $4 + 3\sin\theta$.

Finally students should change the range of T-step to various numbers less than 360° to see partial circles or pieces of circles.

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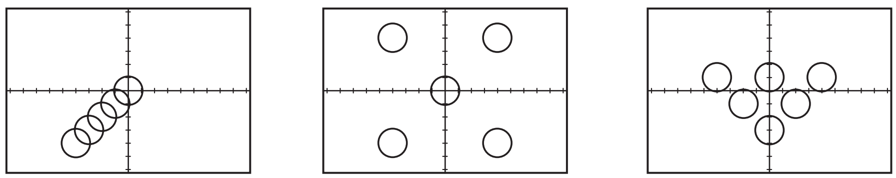
Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

C37(111)

Paper and Pencil

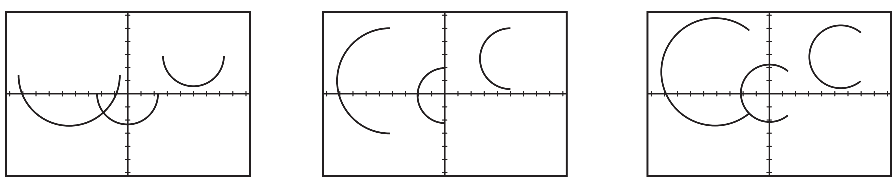
1) Write parametric equations to represent the circles in each of these graphs.



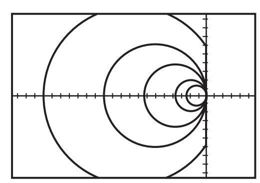
C37(111)

Performance

2) Explain how to get each of these graphs displayed on your screen.



3) This is a design created with parametric equations and a graphing calculator. Create your own design or picture and record the equations, settings, and proper ranges to duplicate it.



4) a) Set maximum T = 180°; use the equations $X = \cos(T)$ and $Y = \sin(T)$.
   b) Set maximum T = 360°; use the equations $X = \cos(0.5T)$ and $Y = \sin(0.5T)$.
   c) Set maximum T = 90°; use the equations $X = \cos(2T)$ and $Y = \sin(2T)$.
   How do these different descriptions affect the way the graphing calculator plots the points? Which is the fastest? Which is the most accurate? Explain your answer.

Suggested Resources

Appendix B:
SCOs for Grades 9 and 10

GCO A: Students will demonstrate number sense and apply number theory concepts.

Elaboration: Number sense includes understanding of number meanings, developing multiple relationships among numbers, recognizing the relative magnitudes of numbers, knowing the relative effect of operating on numbers, and developing referents for measurement. Number theory concepts include such number principles as laws (e.g., commutative and distributive), factors and primes, number system characteristics (e.g., density), etc.

By the end of grade 9, students will be expected to

A1 solve problems involving square root and principal square root

A2 graph, and write in symbols and in words, the solution set for equations and inequalities involving integers and other real numbers

A3 demonstrate an understanding of the meaning and uses of irrational numbers

A4 demonstrate an understanding of the inter-relationships of subsets of real numbers

A5 compare and order real numbers

A6 represent problem situations using matrices

By the end of grade 10, students will be expected to

A1 relate sets of numbers to solutions of inequalities

A2 analyse graphs or charts of situations to derive specific information

A3 demonstrate an understanding of the role of irrational numbers in applications

A4 approximate square roots

A5 demonstrate an understanding of the zero product property and its relationship to solving equations by factoring

A6 apply properties of numbers when operating upon expressions and equations

A7 demonstrate and apply an understanding of discrete and continuous number systems

A8 demonstrate an understanding of and apply properties to operations involving square roots

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B1 model, solve, and create problems involving real numbers

B2 add, subtract, multiply, and divide rational numbers in fractional and decimal forms, using the most appropriate method

B3 apply the order of operations in rational number computations

B4 demonstrate an understanding of, and apply the exponent laws, for integral exponents

B5 model, solve, and create problems involving numbers expressed in scientific notation

B6 judge the reasonableness of results in problem situations involving square roots, rational numbers, and numbers written in scientific notation

B7 model, solve, and create problems involving the matrix operations of addition, subtraction, and scalar multiplication

B8 add and subtract polynomial expressions symbolically to solve problems

B9 factor algebraic expressions with common monomial factors concretely, pictorially, and symbolically

B10 recognize that the dimensions of a rectangular model of a polynomial are its factors

B11 find products of two monomials, a monomial and a polynomial, and two binomials, concretely, pictorially, and symbolically

By the end of grade 10, students will be expected to

B1 model (with concrete materials and pictorial representations) and express the relationships between arithmetic operations and operations on algebraic expressions and equations

B2 develop algorithms and perform operations on irrational numbers

B3 use concrete materials, pictorial representations, and algebraic symbolism to perform operations on polynomials

B4 identify and calculate the maximum and/or minimum values in a linear programming model

B5 develop, analyse, and apply procedures for matrix multiplication

B6 solve network problems using matrices

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B12 find quotients of polynomials with monomial divisors

B13 evaluate polynomial expressions

B14 demonstrate an understanding of the applicability of commutative, associative, distributive, identity, and inverse properties to operations involving algebraic expressions

B15 select and use appropriate strategies in problem situations

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally.

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 9, students will be expected to

C1 represent patterns and relationships in a variety of formats and use these representations to predict and justify unknown values

C2 interpret graphs that represent linear and non - linear data

C3 construct and analyse tables and graphs to describe how changes in one quantity affect a related quantity

C4 determine the equations of lines by obtaining their slopes and y-intercepts from graphs, and sketch graphs of equations using y-intercepts and slopes

C5 explain the connections among different representations of patterns and relationships

C6 solve single-variable equations algebraically, and verify the solutions

C7 solve first-degree single-variable inequalities algebraically, verify the solutions, and display them on number lines

C8 solve, and create problems involving linear equations and inequalities

By the end of grade 10, students will be expected to

C1 express problems in terms of equations and vice versa

C2 model real-world phenomena with linear, quadratic, exponential, and power equations, and linear inequalities

C3 gather data, plot the data using appropriate scales, and demonstrate an understanding of independent and dependent variables, and domain and range

C4 create and analyse scatter plots using appropriate technology

C5 sketch graphs from words, tables, and collected data

C6 apply linear programming to find optimal solutions to real-world problems

C7 model real-world situations with networks and matrices

C8 identify, generalize, and apply patterns

C9 construct and analyse graphs and tables relating two variables

C10 describe real-world relationships depicted by graphs, tables of values, and written descriptions

C11 write an inequality to describe its graph

C12 express and interpret constraints using inequalities

C13 determine the slope and y-intercept of a line from a table of values or a graph

C14 determine the equation of a line using the slope and y-intercept

C15 develop and apply strategies for solving problems

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C16 interpret solutions to equations based on context

C17 solve problems using graphing technology

C18 investigate and find the solution to a problem by graphing two linear equations with and without technology

C19 solve systems of linear equations using substitution and graphing methods

C20 evaluate and interpret non-linear equations using graphing technology

C21 explore and apply functional relationships notation, both formally and informally

C22 analyse and describe transformations of quadratic functions and apply them to absolute value functions

C23 express transformations algebraically and with mapping rules

C24 rearrange equations

C25 solve equations using graphs

C26 solve quadratic equations by factoring

C27 solve linear and simple radical, exponential, and absolute value equations and linear inequalities

C28 explore and describe the dynamics of change depicted in tables and graphs

C29 investigate, and make and test conjectures concerning, the steepness and direction of a line

C30 compare regression models of linear and non-linear functions

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C31 graph equations and inequalities and analyse graphs both with and without graphing technology

C32 determine if a graph is linear by plotting points in a given situation

C33 graph by constructing a table of values, by using graphing technology, and when appropriate, by the slope y-intercept method

C34 investigate and make and test conjectures about the solution to equations and inequalities using graphing technology

C35 expand and factor polynomial expressions using perimeter and area models

C36 explore, determine, and apply relationships between perimeter and area, surface area, and volume

C37 represent network problems using matrices and vice versa

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

Elaboration: Concepts and skills associated with measurement include making direct measurements, using appropriate measurement units and using formulas (e.g., surface area, Pythagorean Theorem) and/or procedures (e.g., proportions) to determine measurements indirectly.

By the end of grade 9, students will be expected to

D1 solve indirect measurement problems by connecting rates and slopes

D2 solve measurement problems involving conversion among SI units

D3 relate the volumes of pyramids and cones to the volumes of corresponding prisms and cylinders

D4 estimate, measure, and calculate dimensions, volumes, and surface areas of pyramids, cones, and spheres in problem situations

D5 demonstrate an understanding of and apply proportions within similar triangles

By the end of grade 10, students will be expected to

D1 determine and apply formulas for perimeter, area, surface area, and volume

D2 apply the properties of similar triangles

D3 relate the trigonometric functions to the ratios in similar right triangles

D4 use calculators to find trigonometric values of angles and angles when trigonometric values are known

D5 apply trigonometric functions to solve problems involving right triangles, including the use of angles of elevation

D6 solve problems involving measurement using bearings and vectors

D7 determine the accuracy and precision of a measurement

D8 solve problems involving similar triangles and right triangles

D9 determine whether differences in repeated measurements are significant or accidental

D10 determine and apply relationships between the perimeters and areas of similar figures, and between the surface areas and volumes of similar solids

D11 explore, discover, and apply properties of maximum areas and volumes

D12 solve problems using the trigonometric ratios

D13 demonstrate an understanding of the concepts of surface area and volume

D14 apply the Pythagorean Theorem

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties, and relationships.

Elaboration: Spatial sense is an intuitive feel for one's surroundings and the objects in them and is characterized by such geometric relationships as (i) the direction, orientation, and perspectives of objects in space; (ii) the relative shapes and sizes of figures and objects; and (iii) how a change in shape relates to a change in size. Geometric concepts, properties, and relationships are illustrated by such examples as the concept of area, the property that a square maximizes area for rectangles of a given perimeter, and the relationships among angles formed by a transversal intersecting parallel lines.

By the end of grade 9, students will be expected to

E1 investigate, and demonstrate an understanding of, the minimum sufficient conditions to produce unique triangles

E2 investigate, and demonstrate an understanding of, the properties of, and the minimum sufficient conditions to guarantee, congruent triangles

E3 make informal deductions, using congruent triangle and angle properties

E4 demonstrate an understanding of and apply the properties of similar triangles

E5 relate congruence and similarity of triangles

E6 use mapping notation to represent transformations of geometric figures, and interpret such notations

E7 analyse and represent combinations of transformations, using mapping notation

E8 investigate, determine, and apply the effects of transformations of geometric figures, on congruence, similarity, and orientation

By the end of grade 10, students will be expected to

E1 explore properties of, and make and test conjectures about 2- and 3-dimensional figures

E2 solve problems involving polygons and polyhedra

E3 construct and apply altitudes, medians, angle bisectors, and perpendicular bisectors to examine their intersection points

E4 apply transformations when solving problems

E5 use transformations to draw graphs

E6 represent network problems as digraphs

E7 demonstrate an understanding of, and write a proof for, the Pythagorean Theorem

E8 use inductive and deductive reasoning when observing patterns, developing properties, and making conjectures

E9 use deductive reasoning and construct logical arguments and be able to determine, when given a logical argument, if it is valid

GCO F: Students will solve problems involving the collection, display and analysis of data.

Elaboration: The collection, display and analysis of data involves (i) attention to sampling procedures and issues, (ii) recording and organizing collected data, (iii) choosing and creating appropriate data displays, (iv) analysing data displays in terms of broad principles (e.g., display bias) and via statistical measures (e.g., mean), and (v) formulating and evaluating statistical arguments.

By the end of grade 9, students will be expected to

F1 determine characteristics of possible relationships shown in scatter plots

F2 sketch lines of best fit and determine their equations

F3 sketch curves of best fit for relationships that appear to be non-linear

F4 select, defend, and use the most appropriate methods for displaying data

F5 draw inferences and make predictions based on data analysis and data displays

F6 demonstrate an understanding of the role of data management in society

F7 evaluate arguments and interpretations that are based on data analysis

By the end of grade 10, students will be expected to

F1 design and conduct experiments using statistical methods and scientific inquiry

F2 demonstrate an understanding of the concerns and issues that pertain to the collection of data

F3 construct various displays of data

F4 calculate various statistics using appropriate technology, analyse and interpret data displays, and describe relationships

F5 analyse statistical summaries, draw conclusions, and communicate results about distributions of data

F6 solve problems by modeling real-world phenomena

F7 explore non-linear data using power and exponential regression to find a curve of best fit

F8 determine and apply the line of best fit using the least squares method and median-median method with and without technology, and describe the differences between the two methods

F9 demonstrate an intuitive understanding of correlation

F10 use interpolation, extrapolation and equations to predict and solve problems

F11 describe real-world relationships depicted by graphs and tables of values

F12 explore measurement issues using the normal curve

F13 calculate and apply mean and standard deviation using technology, to determine if a variation makes a difference

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration: Representing and solving problems involving uncertainty entails (i) determining probabilities by conducting experiments and/or making theoretical calculations, (ii) designing simulations to determine probabilities in situations which do not lend themselves to direct experiment, and (iii) analysing problem situations to decide how best to determine probabilities.

In grade 9, students will be expected to

By the end of grade 9, students will be expected to

G1 make predictions of probabilities involving dependent and independent events by designing and conducting experiments and simulations

G2 determine theoretical probabilities of independent and dependent events

G3 demonstrate an understanding of how experimental and theoretical probabilities are related

G4 recognize and explain why decisions based on probabilities may be combinations of theoretical calculations, experimental results, and subjective judgements